In the Superior Court of Pennsylvania

Nos. 167 & 168 MDA 2009

COMMONWEALTH OF PENNSYLVANIA,

Appellant,

v.

DIANE ALICE DENT,

Appellee.

COMMONWEALTH OF PENNSYLVANIA,

Appellant,

v.

WALTER LEROY WATKINS,

Appellee.

BRIEF OF THE POKER PLAYERS ALLIANCE AS *AMICUS CURIAE* IN SUPPORT OF APPELLEES

On Appeal from the Judgments of the Court of Common Pleas of Columbia County, entered January 14, 2009 at Nos. 733 and 746 of 2008

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STATEMENT OF INTEREST

Amicus curiae the Poker Players Alliance is a nonprofit organization whose members are poker players and enthusiasts from around the United States, including thousands in the State of Pennsylvania. The Alliance works to protect the legal rights of poker players. The group's membership has a direct interest in the outcome of this case, because the Commonwealth's theory would preclude many innocent individuals from playing poker in traditional social situations that the state legislature never intended.

ARGUMENT

Defendants-Appellees in this case were charged with allowing persons to assemble for the purpose of unlawful gambling and related charges under 18 PA. CONS. STAT. ANN. §§ 306(1)(i)(ii) and 5513 for hosting a game of the most popular form of poker –Texas Hold 'Em. Appellees preemptively moved for habeas corpus relief on the ground that Texas Hold 'Em does not constitute unlawful gambling. Appellees and the Commonwealth agreed that the Commonwealth must establish three elements in order to sustain a charge of illegal gambling under Pennsylvania law: consideration, reward, and chance. The first two elements were not contested. The sole disputed issue was whether Texas Hold 'Em is a game of chance.

Chance is often a question of degree. The habeas court concluded that the "controlling" question was "whether Texas Hold'em is a game of skill or a game of chance or, if both, does skill trump chance or vice-verse," such that "if chance predominates, Texas Hold'em is gambling" while "[i]f skill predominates, it is not gambling." Op. at 4. The parties agreed. The Commonwealth described the proper legal test under Pennsylvania law as follows: "The question is, does skill *predominate* o[ve]r chance or chance predominate over skill? If skill predominates over chance, it is not illegal gambling." Tr. at 21 (emphasis added). Appellees concurred. *Id.*

The habeas court found that "Texas Hold'em poker is a game where skill predominates over chance" and so is not unlawful gambling under the Crimes Code. Op. at 14. The court reached that conclusion after considering the holding of *Commonwealth v. Two Electronic Poker Game Machines*, 502 Pa. 186, 465 A.2d 973 (1983),[1] that certain simple machine-based versions of poker are unlawful gambling because, in part, skill played a significantly lesser role in the game at issue than it does in poker games between human players. *See id.*, 502 Pa. at 196, 465 A.2d at 978 ("That the skill involved in Electro-Sport is not the same skill which can indeed determine the outcome in a game of poker between human players can be appreciated when it is realized that holding, folding, bluffing and raising have no role to play in Electro-Sport poker."). The habeas court also took into account an array of scholarly articles analyzing poker from legal, mathematical, and statistical perspectives.

The Commonwealth appealed.

I. WHETHER PLAYING TEXAS HOLD 'EM IS "UNLAWFUL GAMBLING" WAS AN OPEN QUESTION, AND THE COMMONWEALTH'S ARGUMENT THAT THE CASES FORECLOSED THAT INQUIRY WAS WAIVED

On appeal, the Commonwealth contends that prior rulings "unequivocally mandate" the conclusion that Texas Hold 'Em is "unlawful gambling." Appellant's Br. at 10. It neglects to mention that it described the same case law to the habeas court as making "interesting reading" (Tr. at 28), but as "nebulous" (*id.* at 27) and "not terribly helpful" (*id.* at 28). As Appellees explain (Appellees' Br. at 13-14), the Commonwealth's argument is waived, because its only argument below was that, as a matter of fact, chance predominates over skill in Texas Hold 'Em. *See*

[1] That decision resolved the consolidated cases of *Commonwealth v. Two Electronic Poker Game Machines*, *Commonwealth v. One Electronic Poker Game Machine* and *Commonwealth v. One Electro-Sport Draw Poker Machine*. The habeas court referred to the decision as *One Electro-Sport Draw Poker Machine* because it relied on the portion of the opinion that resolved the issues specific to that case. *Amicus* refers to the decision as *Two Electronic Poker Game Machines* because that is how the Commonwealth refers to it on appeal.

Pa. R. App. P. 302(a) (issues not raised below may not be raised on appeal). The Commonwealth "cannot be heard to complain" that the habeas court made the very inquiry into the role of skill in poker that invited: if it was error, that "error was of his own making." *Kriner v. Dinger*, 297 Pa. 576, 582, 147 A. 830, 832 (1929).

Appellees are further correct that the Commonwealth misreads the case law which it cites. The decision on which it principally relies, *Pennsylvania Liquor Control Board v. Kehler*, 538 A.2d 979 (Pa. Commw. Ct. 1988), is not relevant because it arises in the context of the Liquor Code, not the Crimes Code's prohibition on unlawful gambling. A liquor license may be revoked if the establishment permits even *lawful* gambling, and gambling in the liquor licensing context is furthermore defined without regard to whether skill predominates over chance. *Id.* at 980-81 (citing *Pennsylvania Liquor Control Bd. v. PPC Circus Bar, Inc.*, 506 A.2d 521 (Pa. Commw. Ct. 1986)). *Kehler* did not apply that criminal standard—indeed it expressly declined to address whether poker is "unlawful gambling" under § 5513 of the Criminal Code. *Kehler*, 538 A.2d at 981 ("we are not prepared to hold and need not decide that poker playing is 'unlawful gambling' under the Crimes Code").

The legal standard in the criminal context is very different. In determining whether an activity constitutes "unlawful gambling" under the Crimes Code, the question is whether skill or instead chance predominates. As noted, the Commonwealth conceded below that was the correct legal standard. That concession was correct under the governing case law. *Two Elec. Poker Game Machs.*, 502 Pa. at 195, 465 A.2d at 978 ("While appellee has demonstrated that some skill is involved in the playing of Electro-Sport, we believe that the element of chance predominates and the outcome is largely determined by chance.").

For these reasons, the habeas court was correct to decide the factual question whether skill or chance predominates in Texas Hold 'Em, and the Commonwealth is in any event barred from raising a claim of error on that ground.

II. SKILL PREDOMINATES OVER CHANCE IN TEXAS HOLD 'EM

What remains is the question whether this court should review the habeas court's factual determination that skill predominates over chance in Texas Hold 'Em. The government does not dispute that the court's holding that skill predominates was correct as a general matter, but argues that chance predominated over skill in "Texas Hold 'em as played in this case." As Appellees note, this argument was not raised at the hearing and so is also waived. Appellees' Br. at 17-19.

Should this Court decide to reach the Commonwealth's argument, however, it should reject it on the merits. As *amicus* explains below, skill predominates over chance in Texas Hold 'Em. And since the government offered no evidence that the amount of skill involved in Texas Hold 'Em "as played in this case" differs from the amount involved when that game is played in general, the habeas court's conclusion that Appellees' activity was not "unlawful gambling" was correct.

As is true for similar games like golf, billiards, and bridge, when good poker players play against bad players, they consistently beat them. Players who enter golf and bridge tournaments pay a fee to enter, and earn a cash reward if they win, but these games are contests of skill because their outcome is determined principally by skill. *See Two Elec. Poker Game Machs.*, 502 Pa. at 195, 465 A.2d at 977 ("[i]t cannot be disputed that football, baseball and golf require substantial skill, training and finesse" even though "the result of each game turns in part upon luck or chance"); *In re Allen*, 377 P.2d 280, 281 (Cal. 1962) (bridge requires skill and is not a "game of chance"). So too with poker. To be sure, there is some cumulation of luck over the course of a poker match that will affect how individual players perform. That is also true, for example, of golf, where "changes in the weather may produce harder greens and more head winds for the tournament leader than for his closest pursuers" or a "lucky bounce may save a shot or two." *PGA Tour, Inc. v. Martin*, 532 U.S. 661, 687 (2001). But, as in golf, skill is nonetheless dominant in poker play. The fact that every hand of poker involves multiple decision points (at each of the multiple rounds of betting), multiple decisions at each decision point (bet, call, raise, or fold), and innumerable factors that call for skill to evaluate each of those decisions (for example, the player's own cards, the odds of his hand improving, his sense of the strength of the other player's hand, his sense of the other players' perception of him), establishes that poker is a contest of skill.

Where a game is one which contains elements of both skill and chance, such as poker, two general methods of determining the predominant element have developed. The traditional method courts have used to determine whether a game is predominantly one of skill is to evaluate the game's structure and rules. If the structure and rules allow sufficient room for a player's exercise of skill to overcome the chance element in the game, the game is one of skill and the gambling laws do not apply. *See, e.g.*, *In re Allen*, 377 P.2d at 281-82 (holding the card game of bridge to be one predominantly of skill). A second approach, more favored by the scientific community, is an empirical approach that examines the actual play of the game. It being well accepted that in a game predominated by skill the more skillful players will consistently perform better (*see, e.g.*, Patrick Larkey *et al.*, *Skill in Games*, 43 MANAGEMENT SCIENCE 596 (May 1997)), this approach looks for specific instances over repeated trials to see if in fact the "more skillful players tend to score better than less skillful players." *Id.* at 596. Using either method confirms that the game of poker is predominantly a game of skill.

This section thus proceeds first with an analysis of the structure of poker, demonstrating the importance of making correct decisions in poker, and the degree of skill required to make those decisions. *See* § II.A. It shows that how a person plays his cards is far more important to the person's winning or losing than what actual cards the person is dealt. The subsequent part of this section lays out the results of recent scientific studies based on computer simulations of poker matches and statistical analysis of the actual results of poker matches, all of which demonstrate that more skillful players consistently outperform less skillful players. *See* § II.B.

A. Making Correct Decisions In Poker Requires A Diverse Array Of Sophisticated Skills That Games Of Chance Do Not

The essence of poker is correct decision-making. Each time it is a player's turn to act, he must choose among several decisions, typically whether to bet, raise, or fold. During the course of a single session, a player will have to make hundreds of those decisions. Each time, in order to make the optimal decision the player must take into account a variety of known and unknown factors. The importance of decision-making in poker cannot be understated: in a recent statistical analysis of millions of actual poker hands, the players' decisions *alone* rather than the cards dealt accounted for the result in 76% of all the hands played. *See* Paco Hope, Brian Mizelle & Sean McCulloch, *Statistical Analysis of Texas Hold'Em* at 5 (Jan. 28, 2009) (attached for the Court's convenience as Ex. A).[2] In other words, in those 76% of hands, all but one player folded, making the remaining player the hand's winner, and the actual cards were never revealed. With player decisions deciding more than three-quarters of all poker hands, the players who consistently make good decisions will win. Those who do not will lose.

[2] http://www.cigital.com/resources/gaming/poker/100M-Hand-AnalysisReport.pdf.

To make the right decisions consistently, poker players must employ a range of skills. By skill, *amicus* does not mean simply a sophisticated knowledge of odds, which is merely a prerequisite to competent poker play. To be skilled at poker, players must develop an ability to directly influence the way an individual hand turns out—who collects the pot at the end, and how much is in the pot. As the court below held, "[s]uccessful players must possess intellectual and psychological skills. They must know the rules and the mathematical odds. They must know how to read their opponents' 'tells' and styles. They must know when to hold and fold and raise. They must know how to manage their money." Op. at 13-14.

Of course, it is true that individual moves in poker are called "bets." But that vocabulary is misleading. The "bet" is not a wager on a chance event. Unlike poker "bets," true wagers do not alter the outcome of the event. A bet on the Super Bowl does not change the score; bets at a blackjack table are made before the cards are dealt; bets on roulette wheels are placed before the ball is dropped. Bets at a poker table are different. What is called a "bet" in poker is really a "move" like a move in any other game: it is a gambit designed to provoke a desired reaction from an opponent.

The importance of these moves is heightened because, in typical complex poker games like Texas Hold 'Em, a player must contend with a large number of decision-making stages and a variety of possible courses of action at each stage. In each hand of Hold 'Em, for example, a player has four principal decision-making opportunities: the first after he receives his two down cards, and the next three as the common cards are turned over in three stages. At each stage the player has available to him many courses of action. The focus of each decision is how worthwhile it is to risk additional chips relative to the chance of winning all the chips in the pot in that hand. These decision-making stages reduce the element of chance in the game, since logical decision-making at each of these stages allows the player to control his "fate."

To make optimal moves at each of these stages, players must be mathematicians, observers of human nature, and capable deceivers. Poker players use their "bets" principally to communicate with, manipulate, and intimidate their opponents. Skeptics sometimes say that no amount of skill can turn a deuce into an ace. It is true that skill cannot change the cards, as a great golfer cannot change the wind. But skill allows a poker player with the deuce to make his opponent believe he has an ace, causing his opponent to fold a hand that would have won the pot. So skill also means that a good player will lose less with a deuce and win more with an ace than a bad one. Indeed, as noted, more than 75% of all hands are won when one player bets and all remaining players fold in response. *See* Hope *et al.* at 5; *see also* Howard Lederer, *Why Poker Is a Game of Skill* (May 6, 2008) (unpublished manuscript, attached as Ex. B); World Poker Tour Stats, Website (in World Poker Tour play, only 15% of hands go to a showdown).[3]

Even in that subset of hands, the players typically are not betting on the outcome of a chance event. For example, when a poker player bets as a bluff, he is not hoping that his cards will prove to be better than his opponents'. Instead, the player hopes to win the pot by convincing his opponent to fold the best hand. As it turns out, in roughly 50% of hands that do play to a showdown, a player who would have won had he stayed in will have folded—attesting to the skill of the winning player in scaring his competitor into folding. *See* Paco, *Statistical Analysis* at 5. Of course, a player trying to chase another player out may get called and may lose. But what he was betting on was not what cards his opponents held—the essence of gambling. He was betting to influence what his opponents would do—the essence of strategy.

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[3] http://www.worldpokertour.com/StatsAndTools/Landing.aspx, last accessed Feb. 9, 2009.

The importance of skill in poker is further demonstrated by the fact that a Texas Hold 'Em player who is a beginner can improve his talents and raise the level of his game by study and by accumulating game experience. After only a short time a player can acquire basic game skills, learning when to fold and how to make the basic calculations. But the more a person continues to practice and learn, the more his skills will improve, something which is also true for chess, golf, and bridge players.[4]

Together, the specific skills required to play poker and the demonstrated fact that poker hands are won by maneuvering rather than in a showdown between the dealt cards 75% of the time show that skill is required to be a winning poker player. All of this is particularly true for Texas Hold 'Em, the version of poker played by Appellees here, which is a particularly demanding form of the game and thus distinct from other forms of poker in which skill plays a lesser role. *See* Anthony Cabot & Robert Hannum, *Poker, Public Policy, Law, Mathematics, and the Future of an American Tradition*, 22 T.M. Cooley L. Rev. 443, 483 (skill predominates in Texas Hold 'Em; distinguishing Hold 'Em from other forms of poker); *see also Gallatin County v. D & R Music & Vending, Inc.*, 676 P.2d 779, 781 (Mont. 1984) (in poker "one player pit[s] his skills and talents against those of the other players," distinguishing electronic poker against a machine).

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[4] A significant literature is available to help the novice player develop. *See, e.g.*, Gus Hansen, *Every Hand Revealed* (2008); Daniel Negreanu, *Power Hold'em Strategy* (2008); David Apostolico, *Machiavellian Poker Strategy: How to Play Like a Prince and Rule the Poker Table* (2005); Dan Harrington, *Harrington on Hold 'Em: Expert Strategy for No Limit Tournaments* (2005); Eric Lindgren, *World Poker Tour: Making the Final Table* (2005); Blair Rodman & Lee Nelson, *Kill Phil: The Fast Track to Success in No-Limit Hold 'Em Poker Tournaments* (2005); Doyle Brunson, *Doyle Brunson's Super System: A Course in Power Poker* (2002); David Sklansky, *Tournament Poker for Advanced Players* (2002); David Sklansky, *The Theory of Poker* (1994).

B. Skilled Players Beat Simple Players In Simulated And Real Poker Play

The conclusion that skill is required to win at poker has been further proven by several recent studies. Until quite recently, any rigorous analysis of whether skill or chance predominated in poker could involve only an assessment of the rules of play themselves, because no statistical assessment of the role of skill in poker had been assembled. Now the subject has received academic attention, and the studies uniformly confirm that skill determines the outcome in poker games.

In one recent game-theoretical study, for example, the author demonstrated through the use of a computer simulation that a combination of the skills discussed above is required in order to consistently win at poker. *See* Larkey, *supra.* For his 2001 paper on "Skill in Games," Professor Larkey built a computer model of a simplified version of poker. *See id.* The "general behaviors mandated for player success" at this simplified game were:

- observation,
- memory,
- computation,
- knowledge of the random device,
- misleading opponents about the actual strength of your position, and
- correct interpretation and forecasts of opponents' behaviors.

Id. at 597. To evaluate the relative importance of these areas of skill, singly and in combination, the authors programmed twelve different robot players who would compete against one another. Each was programmed to use a different combination of strategies. *Id.*

The simplest robot only knew the rules of the game—when to bet and how much it was allowed to bet—but aside from that essentially played randomly and without regard to its hand.

A second robot understood the relative values of the hands. It would bet aggressively when it was dealt a good hand, and hold back when it got a bad hand. It ignored its opponents, while three other similar robots made conservative or aggressive assumptions about what the other player's hands contained. Another robot bluffed aggressively.

The more sophisticated robots watched their opponent's betting patterns and made deductions about what those opponents were likely to be holding. Some of these robots would bluff by playing randomly a small percentage of the time in order to confuse other opponents capable of watching and learning.

The authors ran a tournament that pitted each robot player against each other player in 100 one-on-one games.

Over the course of the tournament, the random-play robot won only 0.4% of its games. It lost $546,000. The four robots that dominated the contest were the ones capable of sophisticated calculations about their odds of winning. The robot that could only calculate odds came in fourth. The robot that could calculate odds and that also bluffed occasionally came in third.

But the two most successful robots of all were the robots that most closely emulated real poker players. A robot that not only calculated odds but also observed fellow players and adjusted its style of play came in second at $400,000. The best robot of all calculated odds, learned about its opponents, and bluffed occasionally in order to throw its competitors off track.

Even in the simplified game of poker designed for the study, with simple hands and only two rounds of betting, the best robot was the robot with the essential skills that every poker player learns, practices and tries to master. It calculated the odds it was playing against, which was essential to its success. But it outperformed the others by deceiving its competitors with strategic bluffs while learning about and adjusting to its competitors' style of play. It won 89% of the hands it played, and earned $432,000. *See* Larkey at 601, table 2.

The Larkey study's conclusions are confirmed by several other recent works. Professor Noga Alon at Tel Aviv University created a simplified version of Texas Hold 'Em that examined some elements of basic poker strategy and concluded that "the result of a soccer match, and probably even that of a tennis match, are influenced by chance more than the results in poker played over a long sequence of hands." Noga Alon, *Poker, Chance & Skill* 16–17.[5] "Practice and study do help to improve in poker," he found, and his data supported the conclusion that "skill is far more dominant than luck, and that poker is predominantly a game of skill." *Id.* at 17. Several other scientific papers have similarly concluded that skill predominates over chance in poker play. *See* Laure Elie & Romauld Elie, *Chance and Strategy in Poker* (Sept. 2007) (unpublished manuscript, attached as Ex. C) (examining several variants of Texas Hold 'Em and concluding that skill determined the outcome in all); Abraham J. Wyner, *Chance and Skill in Poker* (Apr. 17, 2008) (unpublished manuscript, attached as Ex. D) (a skilled player who can calculate the odds and bet and bluff on that basis has a substantial advantage over players who lack these skills).

The same conclusion has been reached by two recent legal analyses. Professor Robert Hannum, an statistician who specializes in the mathematics of games, and Anthony Cabot, a leading practitioner of gaming law, ran a sophisticated computer simulation and concluded that poker victory depends on skill. Cabot, *Poker* (conducting Texas Hold 'Em simulations to determine that skilled opponents beat unskilled ones). The author of a second recent article similarly concluded that "poker should not constitute a 'game subject to chance.'" Michael A. Tselnik,

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[5] www.math.tau.ac.il/~nogaa/PDFS/skill4.pdf, last accessed Feb. 9, 2009.

Check, Raise, or Fold: Poker and the Unlawful Internet Gambling Enforcement Act, 35 Hofstra L. Rev. 1617, 1664-65 (Spring 2007).

The number of identifiable skills required to excel at poker and the simulations and studies just discussed all predict that, in real life, the more skilled players will win. In fact, that is what we find. The best players beat other players as often as the best golfers beat other golfers. The fact that poker has a "random device" (*see* Larkey at 597) introduces short term uncertainty into each hand, but over time the randomness of the cards evens out and all players eventually get the same share of good and bad hands. Their results differ based on how skillfully they play those hands.

A striking example of the limited role that the cards play in determining the outcome of poker matches may be found in the recent story of Annette Obrestad, a 19-yead-old poker prodigy who beat 179 other players—without looking at her own cards. *See* Shawn Patrick Green, *Online Poker: Interview With Annette 'Annette_15' Obrestad*, CardPlayer.com (Aug. 12, 2007).[6] Obrestad's feat shows it is the player's skill rather than the deal of the cards that determines the outcome of poker play.

The same result is demonstrated by comparing the results of recent golf and poker tournaments. In the 25-year period beginning with 1976 and ending in 2000, 21 different players won the World Series of Poker. One person won three times in that span (Stu

	Poker	**Golf**
Back-to-back winners	3	2
Number of different winners	21	22
Winners of more than 1 title	4	3
Winners with other top 10 finishes	14 of 21	15 of 22
Top 10 finishes per winner	2.48	2.96
Average number of entrants	187	156

[6] http://www.cardplayer.com/poker-news/article/2536/online-poker-interview-with-annette-annette-15-obrestad, last accessed Feb. 9, 2009.

Ungar), and three more won twice (Johnny Moss, Doyle Brunson and Johnny Chan). Three of these repeat winners won back-to-back wins in consecutive years (Brunson, Ungar and Chan). Fourteen of the twenty-one were "repeat finalists" who finished among the top ten in one or more of the other tournaments.

In the same period, there were twenty-two different winners of the PGA Championship, and three multiple winners. Only Tiger Woods won back-to-back titles. Fifteen of the twenty-two champions made it into the top ten in another Championship. These numbers confirm that poker requires as much skill as golf to win consistently. *Accord* Rachel Croson, Peter Fishman & Devin G. Pope, *Poker Superstars: Skill or Luck?* CHANCE (Vol. 21, No. 4, 2008) (concluding that golf and poker require similar amount of skill to win).

It is precisely because poker requires roughly the same amount of skill as golf that poker tournaments now rival golf tournaments in popularity on television. The only people who watch anyone play roulette on television are casino security guards. People only watch lottery drawings to see if they have won. But poker matches are spectator events because, as in any game that people tune in to watch, it is fun to watch good players get beaten by even better players. Like golf, poker is a game won and lost predominately on the basis of the skills of the players. Appellees in this case were playing a game in which skill predominates, and so were not engaged in unlawful gambling.

C. Neither The Hearing Evidence Nor The Materials Cited By The Habeas Court And In The Sections Above Support The Commonwealth's Contention That Texas Hold 'Em As Played Here Was Gambling

In the face of the authority cited above and in support of the habeas court's decision, the Commonwealth does not dispute that Texas Hold 'Em is a game of skill in general. But it suggests that playing the game constituted unlawful gambling under the conditions in which the game was played in this case. The Commonwealth's own evidence established that Appellees played Texas Hold 'Em, the poker variant described in the books and studies just cited, and indicated that on each occasion the players played multiple hands. *See, e.g.*, Tr. at 6 ("Q: How about the others at the table, do they make an initial bet [i.e., in each hand]? A: No, but it progresses so eventually they all will."). That is typical of poker play. *See* Alon at 18 ("common practice is to play many hands"). But the Commonwealth now suggests that the playing time here was not "sufficiently long" so that the initial cards would "necessarily even out" and that the Texas Hold 'Em play in this case was therefore unlawful gambling. This contention fails for five reasons.

First, the Commonwealth's position would mean that playing Texas Hold 'Em is not unlawful gambling if the players play together long enough, but that if they stop playing too soon, what they were doing will have been unlawful. The law cannot draw such fine lines because a rule like the Commonwealth's would not give individuals the notice they need of whether they activity is permitted or prohibited. As the California Supreme Court held in *In re Allen*, 377 P.2d 280, 281 (Cal. 1962), the focus of the law is on the "character of the game" itself—it is not on the circumstances under which it is played. In granting a habeas petition on the ground that bridge was not prohibited for being a game of chance, the court examined the rules of bridge (*id.* at 281-82) and the literature evaluating the skill involved in playing bridge (*id.* at 282), rather than looking to the way the game was played in any particular case. *See also Morrow v. State*, 511 P.2d 127, 129 (Alaska 1973) ("It is irrelevant that participants may exercise varying degrees of skill."). The Commonwealth is thus wrong to suggest that the focus is on the circumstances of a particular game. Instead, the habeas court was correct to examine the characteristics of the game of Texas Hold 'Em itself.

Second, the Commonwealth misunderstands the import of statements, such as the one quoted by the habeas court, that chance will "even out" in the long run. Op. at 10. Though a single hand of poker may be determined by chance, it may also be determined by skill. A beginner may play a bad hand and make bad calls and bets but be saved by a miracle card at the end, and one might conclude that such a hand was decided by chance. Alternatively, an experienced player may bluff with a bad hand and induce a worse player to fold the better hand. Such a hand would have been decided by skill. Where a game is of a nature where any single result may be the product of either skill or chance (or a mixture of both) the only way to determine which factor predominates is to evaluate which factor—chance or skill—is producing more results over a reliable sample size. Thus a statement like "in the long run skill will prevail at Texas Hold 'em poker" does not mean that a "long run" is *necessary* for skill to prevail. The point of referring to a "long run" is to recognize merely that to reach a reliable conclusion as to which factor is predominant, one must examine a statistically significant sample size of poker hands. As the trial court correctly noted, every published study and learned treatise that has done that has concluded that skill is the predominant element.

Third, and as described above, the Commonwealth's position misunderstands the applicable standard. It is acknowledged that chance plays a role in poker because the deal of the cards is random. The question, as the Commonwealth and the habeas court acknowledged, is whether chance or skill predominates. The Commonwealth's suggestion that Texas Hold 'Em is gambling unless the players play long enough for the role of chance to "even out" as a statistical matter is a suggestion that chance must be statistically eliminated from the game. But the standard is predominance, and in even a limited number of hands, skill predominates over chance. As Professor Alon concluded in the study cited above, in even a single hand of poker the role of chance appears to be smaller than the role of chance in a point in tennis, and "the significance of skill increases dramatically as the number of hands played grows" such that "skill is far more dominant than luck" in any significant number of hands. *Alon* at 14, 16, 18. The habeas court thus properly held that Texas Hold 'Em is not unlawful gambling because the skill involved predominates over chance, not that chance would play no role in the outcome over the course of play.

Fourth, there is no basis in the record for the contention that skill is no longer a factor when poker is played "by shifting casts of characters." As the evidence discussed above demonstrates, the skills involved in poker apply in any context, against any group of players. For example, in the real poker tournaments in § I.B, the winning players had to beat out table after table of competitors in order to make it to the final table—such tournaments proceed by eliminating weaker players who lose all of the their chips and repeatedly consolidating the remaining players into fewer and fewer tables. Good players demonstrate their skill by beating players they have never played against before.

Fifth, the Commonwealth's position ignores the role of skill even in a single hand. As discussed, in 76% of hands, one player wins by convincing the other players that they should fold. Furthermore, a player's assessment of his own cards and what cards the other players are holding will affect whether and how much the player bets. So even in the 24% of hands that reach a showdown, in which the cards dealt will determine who wins the pot, the players' skill will determine how much is won. And as it is the amount won, not who won, that is the outcome of a poker hand, skill is the predominant factor in determining even the outcome of a single hand of poker.

For all of these reasons, even if the Commonwealth had raised its argument below, that argument would fail on the merits.

CONCLUSION

For the foregoing reasons, the Court should hold that the habeas court did not err in holding that § 5513 allows playing Texas Hold 'Em, and should dismiss the Commonwealth's appeal.

Respectfully submitted this 5th day of June, 2009.

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CERTIFICATE OF SERVICE

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EXHIBIT A

cigital

Statistical Analysis of Texas Hold'Em

January 28, 2009

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This work was performed under contract to:
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Douglas Bay Complex, Onchan
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Note to the Reader

This document is written to a technical audience. It is assumed that the reader is acquainted with common poker terminology (flop, river, hole cards, board, etc.) It is further assumed that the reader understands the basic mechanics of playing Texas Hold 'Em. This document also uses standard poker notation such as K♦4♣Q♦2♠J♥ or 5c5hKcTd8d to represent hands.

1 Executive Summary

The effect of luck (i.e., the dealing of the cards) in Texas Hold'Em is a subject of much debate in the legal community. This study seeks to establish clear numbers derived from a significant sample of actual play. This study does not quantify the effect that luck has on Texas Hold'Em, but it provides compelling statistics about the way that the outcomes of games are largely determined by players' decisions rather than chance.

Cigital examined 103 million hands of Texas Hold'Em poker played at PokerStars. In the majority of cases, 75.7% of the time, the game's outcome is determined with no player seeing more than his/her own cards and some or all of the community cards. In these games all players fold to a single remaining player who wins the pot. In the 24.3% of cases that see a showdown (where cards are revealed to determine a winner), only 50.3% of showdowns are won by the player who could make the best 5-card hand. The other roughly half of the showdowns are won by someone with an inferior 5-card hand because the player with the best 5-card hand folded prior to showdown.

We use accepted statistical sampling formulas to make the argument that these statistics are generally representative of Texas Hold'Em in Section 2. The raw findings themselves are presented in Section 3. In order that the artifacts can be reused with confidence, the cryptographic signatures of all contributing data are listed in Section 5.

2 Goals and Methodology

The purpose of this analysis is to determine certain statistical qualities of the game of Texas Hold 'Em as played at PokerStars.com. Given the specific results from analyzing PokerStars.com, we want to generalize the results and say mathematically that they represent the game of Texas Hold 'Em as a whole. It is important that Cigital conduct this analysis independently and without predisposition towards the final outcome.

2.1 Data Acquisition

Cigital acquired data from REEL related to play at PokerStars.com. The log files are archived by Cigital and their SHA-1 signatures are recorded in Section 5. The log files contain descriptions of the play of many hands. Table 1 shows two groups of log file lines that describe two different games. Note that user IDs have been changed and the hand IDs are fictitious to protect the confidentiality of this data.

Game	Blind	Bet	Hand ID	Board	User ID	Pos	Win	Hole	Best Hand	Show
No Limit	100	200	1399167686	8dKcTd9sQd	Player A	0	0	KsQh	KsKcQhQdTd	1
No Limit	100	200	1399167686		Player B	1	0	2s7s	7s2s	0
No Limit	100	200	1399167686	8dKcTd9sQd	Player C	2	1	4d5d	QdTd8d5d4d	1
No Limit	100	200	1399167686		Player D	3	0	Qc8s	Qc8s	0
No Limit	100	200	1399167686		Player E	4	0	5c5h	5c5hKcTd8d	0
No Limit	100	200	1399167686		Player F	5	0	Tc2d	Tc2d	0
No Limit	100	200	1399167686		Player G	6	0	AsKh	KhKcAsTd8d	0
No Limit	100	200	1399167686		Player H	7	0	3h2c	3h2c	0
No Limit	100	200	1399167686		Player I	8	0	Ah6h	Ah6h	0
No Limit	10	25	1299170765		Player A	0	0	5cQs	5c5sAdQsJh	0
No Limit	10	25	1299170765	9s2d5sAdJh	Player B	1	1	2hTh	2h2dAdJhTh	0
No Limit	10	25	1299170765		Player C	2	0	6c3c	6c3c	0
No Limit	10	25	1299170765		Player D	3	0	3h7s	7s3h	0
No Limit	10	25	1299170765		Player E	4	0	5dTd	Td5d	0
No Limit	10	25	1299170765		Player F	5	0	8c6s	8c6s	0
No Limit	10	25	1299170765		Player G	6	0	3sAc	Ac3s	0
No Limit	10	25	1299170765		Player H	7	0	Kh7c	Kh7c	0
No Limit	10	25	1299170765		Player I	8	0	JsQh	JsJhAdQh9s	0

Table 1: Example Log Data

In the first game, 1399167686, both Player A and Player C went to a showdown. This is indicated both by the fact that the "board" column contains the board next on both players' rows and by the fact that the showdown column is "1." Player C wins with a flush: Q♦T♦8♦5♦4♦ against Player A's two pair.

In the second game, 1299170765, the board is listed next to the singular winner, Player B. In this case, there was no showdown, even though the entire board (all five cards) were dealt. This indicates that all players still in the game when the river was dealt eventually folded to Player B. It is worth noticing that Player B had a pair of 2's as his best hand. Several players (A, G, and I) would have beaten that hand, had they stayed in.

Cigital analyzed 103,273,484 such hands that had the following characteristics:

Cash Ring Games No play money games were considered. No "heads-up" tables were included. That is, there are some two-player games in the sample set, but

	they are situations where two players sat and played against each other at a table that would allow more than two players.
Blinds 10¢ or higher	So-called "microlimit" games (games with blinds less than $1) are considered too much like play money games, so only a few such games (10¢, 25¢, and 50¢) were included. The 2¢ and 5¢ games were excluded.
December 1, 2008 to January 2, 2009	Cigital selected this timeframe because it needed to independently corroborate a subset of the hands played with the actual players themselves. See Section 2.4.

2.2 Data Analysis

For each hand analyzed, two facts were determined:

1. *Did the hand end in a showdown?* A "showdown" is a situation where all four rounds of betting have been completed and more than one player remains in the game. At least one player must show his cards so the winner can be determined.
2. If there was a showdown, *did the player with the best two cards win the hand?* It is relatively common for the best two cards (i.e., the player who would have made the best 5-card hand at showdown) to fold prior to the showdown.

2.2.1 Showdown Determination

Whether or not there is a showdown is a very simple fact to determine. There is no controversy or explanation necessary. Either there was more than one player in the game after all the betting was complete, or there was not.

2.2.2 Best Hand Win Determination

Determining whether the best hand won the showdown requires assumptions to be made. We are considering whether the player whose hole cards would combine with the board to make the best 5-card poker hand was actually the player who won at showdown.

At least two situations arise occasionally that could be considered a best-hand-win or not.

Equivalent Hands: Assume the board is K♦4♣Q♦2♠J♥, and Player A has A♦T♣ and Player B has A♣T♠. Both have an Ace-high straight. Assuming no other players have better hole cards, both Players A and B would win at the showdown and would split the pot. If Player A folds early, but Player B goes on to the showdown, Player B will win the entire pot. It is arguable that since one of the two equivalent hands did go on and win, that the best hand did win this game.

Board Best Hand: In some cases the board is the best hand. For example, if the board is 8♦8♣8♥2♠2♥, it is quite likely (though not certain) that no player has a better hand than a full house 8s full of 2s. In such a situation, where no player's hole cards improve the board, all players who stay to the showdown will split the pot. If one or more players fold before the showdown, they will not share in the pot. This situation is a special case of the "Equivalent Hands" case, because in this situation all players are equivalent. Again, it is arguable that since some hands win at the end, the best hand did win the game.

Cigital has chosen to count both of these situations as hands where the best two cards **did not** win. Since there were players who folded early, but would have been paid had they stayed in, there were "best hands" that did not win. Using the alternative method and not counting such hands would have only a small impact on the final result as such hands are relatively rare.

2.3 Statistical Method

Games in the log data were organized by "game type." Game type is a combination of the game rules (i.e., Limit, No Limit, or Pot Limit), any restrictions on the table size (e.g., 10 players or 6 players) and the blind/bet sizes. For each game type we then performed a statistical analysis of the percentages of showdowns and percentages of showdowns won by the best hand to see how representative they are of Texas Hold 'Em poker hands in general.

2.3.1 Description of the analysis

We are assuming that the distribution of the number of hands that go to showdown and where the best hand won follow the binomial distribution. Specifically, we are treating each hand as a separate independent test, where the results of one hand have no bearing on the results of any other.

When the amount of data is large (as it is in our survey) the distribution of proportions of binomial data fits closely to a normal distribution. This process has several steps:

1) We define X (the number of successes) and N(the sample size). For our purposes, X is the number of hands that went to showdown in the limit we are examining (or, the number of hands where the best hand won). N is the total number of hands surveyed at the limit we're examining.

2) We construct the Wilson Estimate of the proportion:
$$\tilde{p} = \frac{X+2}{N+4}$$
The Wilson estimate is a popular way of adjusting a proportion by acting as if we had two more successes and two more failures. Notice that when the sample size is large (as it is in the majority of our surveys) this adjustment will have almost no effect.

3) We determine the standard error of the proportion (again, assuming that the proportion can be approximated by the normal distribution):
$$SE_{\tilde{p}} = \sqrt{\frac{\tilde{p}(1-\tilde{p})}{n+4}}$$
..which is just the standard deviation under the normal distribution under our Wilson estimate.

4) We then determine a desired confidence level C and determine a confidence interval:
$$\tilde{p} \pm z^{*}SE_{\tilde{p}}$$
where z* is the value for the standard normal density curve with area C between $-z^*$ and z^*. We computed this value for z^* in Microsoft Excel as follows:

    (a) Given the confidence percentage C, we compute the probability of anything being outside of the confidence interval on the right side of the normal distribution by:
    $$p = \frac{1-C}{2}$$

    (b) We then use the Microsoft Excel "NORMSINV" function to find the inverse of the standard normal distribution at probability p. This gives us our z* value. It should be noted that Excel uses an iterative search technique to generate the result, and so the results may not be exactly accurate. However, several checks were made against standard tables and the results of NORMSINV were found accurate to at least three decimal places.

5) Once we have our confidence interval, we can define the margin of error as:
$m = z^{*} SE_{\hat{p}}$

6) If desired, we can also fix a desired margin of error, and compute the required z^{*}(and thus the required confidence level) needed to reach this margin of error by inverting this process.

For the case of determining the number showdowns won by the best hand, we perform the same analysis. We let X represent the number of hands won by the best hand in the limit we are examining. We let N be the total number of showdowns surveyed at that limit.

2.3.2 Assumptions and possible sources of error

As was alluded to above, we made several assumptions during this process. If these assumptions are not valid, that may impact the accuracy of our results.

1) We assume that the data surveyed follows the binomial distribution. Specifically, we assume that each hand is an independent event with fixed probability of a showdown, and that the result of whether one hand went to a showdown has no bearing on whether a subsequent hand goes to showdown.

2) We use the normal distribution to approximate the distribution of the proportions. This is just an approximation, and introduces a potential source of error. However, this is an accepted approximation when $n*p \geq 10$, and $n(1-p) \geq 10$ (where n = the sample size, and p = the proportion of hands that go to showdown), and all of the limits examined are well beyond this lower bound.

3) We assume that December 2008 is a representative month of normal play at PokerStars, and that there is nothing special about it that would cause our extrapolations about how it represents other months in general to be wrong.

4) We assume that the proportions of hands played at the various table types (e.g., $1/$2 No Limit 6 Max) in December are representative of the proportions of play normally. There is nothing special about this sample to cause our extrapolations to be wrong.

5) We assume that the calculations made, both the ones provided by Microsoft Excel functions, and the ones that were made to implement the formulas, are correct. Several entries were checked by hand and found to be correct.

6) We assume that the data collection was accurate, and that PokerStars gave us a complete and accurate representation of all hands played in the requested month, and that the collection of the "number of show-downs" and "total number of hands played" data is correct. Rather than take PokerStars' log files at face value, we performed independent corroboration directly with some players, as described in Section **Error! Reference source not found.**.

2.4 Verifying Log Data

PokerStars players were asked to independently submit their hand histories to Cigital, along with an attestation that the hand history was accurate.

2.4.1 Rationale

Part of the reason that we chose December 2008 as a sample month was so that the players would have their histories fresh. It gave them the best opportunity to honestly recollect their hands.

2.4.2 Mechanics

Each player sent their history by email. It included the following affirmation statement: *I, NAME, affirm that, to the best of my recollection, the attached data is an accurate representation of my activity on PokerStars.com.*

One might dispute the idea that a player can remember 60,000 hands accurately. The players who submitted histories are the kinds of players who use databases while they play. As each hand finishes, it is stored in their personal database. Certainly the player would notice a loss being recorded as a win and such obvious mistakes. The kinds of players who submitted hand histories are diligent and scrupulous about recording and analyzing their play. So, while it is unlikely that they remember all 60,000 hands in mid-January, it is highly likely that they vetted those hands as the hands were added to their database. Furthermore the data the players provided was directly from their private databases, not from PokerStars itself. That is, it was data that they collected prior to our announcement of this study or any request for assistance. Thus, an extraction from their personal databases can be considered independent of PokerStars' influence.

2.4.3 Results

Cigital received six player histories covering 627,314 games. Out of that set of histories, 583,534 applied to our sample set. The other 44,000 hands were either from the wrong date (e.g., November 30) or were from tables we are not analyzing (tournaments, heads-up, low-limit, etc.). This yields 0.56% of hands in the sample data directly confirmed by players. We treat these as samples of log data where a "successful test" is when the player's personal data match PokerStars' log file, and an "unsuccessful test" is when they don't.

All the players' histories agreed with PokerStars log files exactly. We conclude that there is a 99.99% chance that the accuracy of ALL hands is 99.99% ± 0.001%. It is highly improbable that PokerStars modified the data in the log files.

3 Findings

The short summary of our findings is that 24.3% of hands result in a showdown. Of that 24.3% of hands that result in showdown, 50.3% of them are won by all players that were dealt the best two cards initially. Table 2 shows the detailed findings by game type.

Game Rules	Blind	Bet	Percentage Show-downs	Percentage of Showdowns Best-Hand-Win
Limit	10	20	55.6%	52.1%
Limit	25	50	52.4%	49.3%
Limit	50	100	41.7%	44.7%
Limit	100	200	34.0%	42.5%
Limit	200	400	35.2%	43.6%
Limit	300	600	31.9%	43.5%
Limit	500	1000	31.9%	43.1%
Limit	1000	2000	36.3%	47.5%
Limit	1500	3000	37.7%	61.9%
Limit	3000	6000	35.4%	62.8%
Limit	5000	10000	33.3%	71.6%
Limit	10000	20000	35.7%	79.6%
Limit	20000	40000	35.8%	92.3%
Limit	50000	100000	31.1%	98.4%
Limit	100000	200000	26.8%	100.0%
Limit 6 Max	10	20	52.1%	67.6%
Limit 6 Max	25	50	48.1%	64.2%
Limit 6 Max	50	100	43.8%	60.4%
Limit 6 Max	100	200	39.3%	58.4%
Limit 6 Max	200	400	38.3%	58.3%
Limit 6 Max	300	600	34.9%	57.9%
Limit 6 Max	500	1000	32.8%	56.8%

Game Rules	Blind	Bet	Percentage Show-downs	Percentage of Showdowns Best-Hand-Win
Limit 6 Max	1000	2000	34.7%	58.0%
Limit 6 Max	1500	3000	37.1%	61.1%
Limit 6 Max	3000	6000	34.8%	62.5%
Limit 6 Max	5000	10000	35.1%	65.1%
Limit 6 Max	10000	20000	35.6%	72.6%
Limit 6 Max	20000	40000	33.4%	87.5%
No Limit	10	25	26.1%	42.1%
No Limit	25	50	21.1%	39.1%
No Limit	50	100	17.8%	39.0%
No Limit	100	200	14.7%	38.3%
No Limit	200	400	14.7%	41.3%
No Limit	300	600	13.8%	43.5%
No Limit	500	1000	13.6%	43.4%
No Limit	1000	2000	11.6%	60.5%
No Limit	2500	5000	10.8%	69.6%
No Limit	10000	20000	9.9%	90.0%
No Limit 6 Max	10	25	22.0%	52.0%
No Limit 6 Max	25	50	20.0%	50.9%
No Limit 6 Max	50	100	16.1%	50.3%
No Limit 6 Max	100	200	13.8%	50.3%
No Limit 6 Max	200	400	13.3%	51.7%
No Limit 6 Max	300	600	12.4%	53.0%
No Limit 6 Max	500	1000	11.6%	53.6%
No Limit 6 Max	1000	2000	10.8%	60.0%
No Limit 6 Max	2500	5000	9.2%	69.9%
No Limit 6 Max	10000	20000	6.9%	91.2%
No Limit 6 Max	20000	40000	9.2%	100.0%
Pot Limit	10	25	32.3%	46.2%
Pot Limit	25	50	26.9%	43.4%
Pot Limit	50	100	21.8%	41.3%
Pot Limit	100	200	17.7%	41.9%
Pot Limit	200	400	16.7%	49.3%
Pot Limit	300	600	15.5%	52.1%
		Overall	**24.3%**	**50.3%**

Table 2: Detailed Findings[1]

1 Raw numbers of games played and showdowns are not included in this report by request of REEL. REEL considers such detailed play volume to be proprietary information.

3.1 Explanation of Findings

Each column of Table 2 deserves explanation.

Game Type	The game rules, including limits on the number of players at the table.
Blind / Bet	The blind column is the size of the big blind and the minimum pre-flop bet. The bet column is the size of the minimum post-flop bet. Both of these values are expressed in pennies. Thus a game with 2500 in the blind column and 5000 in the bet column is commonly notated a $25/$50 game.
Percentage of Showdowns	The number of showdowns that were seen at the given game type and bet limits is divided into the total number of hands played at that game type and limit.
Percentage of Best Hand Win	The number of games won by the best hand is divided by the total number of showdowns (not total number of hands) to determine what percentage of showdowns are won by the player who had the best two cards. The determination of best two cards is described above.

3.2 Margin of Error

To calculate the margin of error, we assumed a confidence level of 99.99%. The margin of error for the calculation of showdowns is estimated at ± 0.02%. The margin of error for the calculation of best hands winning is estimated at ± 0.01%. Individually, all but eight of the 55 game types had margins of error < ± 1%. Those eight game types did not experience significant play volume in the sample.

To explain the effect of margin of error, consider a specific game-type: Limit 10¢/20¢ in December 2008. 55.6% of those hands went to showdown that month at that limit. If we were to sample lots and lots of months, we would expect some months to have a higher percentage, some months to have a lower percentage, and so on. These different percentages would stack up in a normal distribution (the bell curve, see Figure 1) **assuming that there is no reason for there to be differences in the data, other than random chance**.

That final assumption is critical. We can only extrapolate these values to be representative of reality if we assume that December 2008 is representative of reality.

Since the samples of all of the months fall into a normal distribution, we need to determine what the odds are that example month falls into the "fat" part of the bell curve. That's where confidence intervals and margins of error come into play.

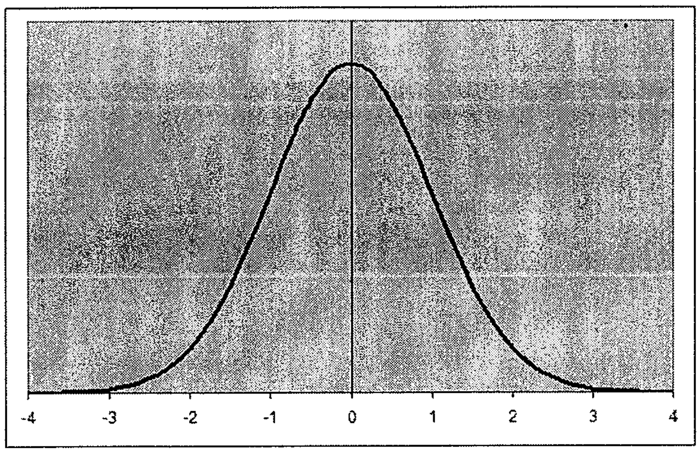

Figure 1: Standard Bell Curve

Figure 1 is a "standard" distribution, which means that it has been rescaled to be centered around 0.

Given that 55.6% of the hands went to showdown. We want to know how likely it is that the "real" bell curve for this situation has its center at, or close to, 55.6 (in other words, how likely is it that the "0" position in the picture is really at 55.6?). Obviously, it is unlikely that it will be exactly 55.6%, but the margin of error gives us a range. If we set the margin of error to 0.1% in the calculations we are asking *How likely is it that the center is 55.6%, ± 0.1%?* It's never a sure thing—it's always theoretically possible that we had a freakishly weird month, but the more hands we sample, the less likely that's true. This is just like it's not too hard to have 9 out of 10 coin flips come up heads, but it's really unlikely—though theoretically possible—to have a 90% heads rate after a million coin flips. The confidence interval comes out to about 99%, and it's based on the margin of error we set. So, what that means is that it is 99% likely that the "0" position of the bell curve in our situation is between 55.5% and 55.7%.

If we increase the margin of error, our confidence goes up (because we have a wider range to cover, so it's more likely that the real center is in that range). If we decrease the margin of error, our confidence goes down (for the same reason).

We can also perform this calculation in the reverse direction. Suppose we want to have a certain confidence that the results are not a fluke. How wide a margin of error do we need for it to be that likely? If we work in this direction and look for a confidence level of 99.99%, we figure out how wide a band of possibility is needed to be 99.99% likely that the "0" position of the real distribution is within that band, based on our estimate. It turns out to be 0.24%. In other words, we believe it is 99.99% likely that 55.6% ± 0.24% of hands at the 10¢/20¢ limit will end up in a showdown.

4 Conclusion

It is clear from these numbers that, at least in the sampled data, the majority of games are determined by something other than the value of the cards, since no player reveals any cards to determine the winner. Only rarely (about 12% of all hands) does the player with the best initial hand go all the way to showdown and win. The statistical analysis of the logs gives us confidence that the logs accurately describe what was played. The analysis of the hands gives us confidence that this sample represents online Texas Hold'Em at PokerStars as a whole.

5 Recorded Artifacts

The following log files and hand histories were received, stored, and used for this analysis.

5.1 PokerStars Log Files

File	SHA-1 signature	File	SHA-1 signature
HandsDec01.txt.gz	c5501596528dc717338b2a53c0d224c125d79729	HandsDec17.txt.gz	e0f82db68d4411724a45b5c383ff8e0ebf790a58
HandsDec02.txt.gz	90caeb2cbda43c7720d628bb3f92d731b7128ad9	HandsDec18.txt.gz	6f4d4209b78bdcf0a7486fea5e92b7d4678e3123
HandsDec03.txt.gz	cf3aac342ded4951d550090d4dcf05bc77ca633a	HandsDec19.txt.gz	4bd8bdf4e28b01d10a94e87d93d631f7f36b8c15
HandsDec04.txt.gz	b8d4c3dc5301384fd7e9da6210c0f04ed248aa98	HandsDec20.txt.gz	a318b050d9f4c019531fe1295c334bb1aa6cc68b
HandsDec05.txt.gz	717d0d87cd7d290533f3b70a9e9cb8b5f0bf7f6e	HandsDec21.txt.gz	b3920863256aa224831eebeaf93cf1145f6435ca
HandsDec06.txt.gz	8150330d3b7eb38af78c83ed6c0a3a45c197e216	HandsDec22.txt.gz	eaa2fdec8512a2cef09c89188600640e68cfca24
HandsDec07.txt.gz	2289a717c1896468d069b6331e96a4197317d446	HandsDec23.txt.gz	623c5a6e5021e1560cbfcbe506c8cf7fe40af8c6
HandsDec08.txt.gz	641ffb8ed18a27d17fd7ba7d25646257cf7343ac	HandsDec24.txt.gz	524c35fb57532166bf684f6ac0f64bd0e1c76093
HandsDec09.txt.gz	bfb86ba566571a2b5fb5b2d3cd8bc97770c2bfc5	HandsDec25.txt.gz	1996e0479bb2e8bc5557578c13d3ea4b591639f5

HandsDec10.txt.gz	20f27406f47b080cb0cd09112dde2f52deb96453	HandsDec26.txt.gz	14e1c82537b2a1c88bae32e4fbc53f738cbe4ef5
HandsDec11.txt.gz	1fb1d1ade45fd2b649e055956494ca207e076bf8	HandsDec27.txt.gz	d0d13614584ab7e6b335df8f402e6d8c94b309a5
HandsDec12.txt.gz	3aee3fd7a538096104ffbf22a9f44b010beb13b7	HandsDec28.txt.gz	7373859b2120dc6681b9d382abd0c7dedde9bb3b
HandsDec13.txt.gz	2dc2b691fc6559ea5f0d3553616ebcad1a96529e	HandsDec29.txt.gz	d901cdc805c2fed8561f119139503b5e187f03a6
HandsDec14.txt.gz	df5f318f3b0f97f49a65369a1d849109c2a572f4	HandsDec30.txt.gz	44214e493fdaf335aa019077b7066c2254650597
HandsDec15.txt.gz	5ec47e468f03c51ac6637c2d567806ed370200f4	HandsDec31.txt.gz	19ec3cdfa2beddeb2bf39a81a5d62871e732877c
HandsDec16.txt.gz	d1384390abae8ec2c927892a364bd78b0ffc45c6	HandsJan01.txt.gz	b5ee0ff2401ef9c03551159f45244a8ad2368bc1
		HandsJan02.txt.gz	94e55df1892c64bfa7a4e7a804b6bd4ee5f891cc

5.2 Player-submitted Hand Histories

SHA-1 Signature	Archive File
f620fad11de3347002f76b680bc215469d4236c9	furbean.zip
5588409225a4a09482008301e21a72d37731df01	LihanLi.zip
621d2508b6836fce55169acc5d344e9b3e1e47bb	basile.zip
9b6ed3073b4bc4823f7fe274b255ee5c6b9728b8	buntaine.zip
0cee4d4e03cb472d08bbb9674fb8c4504e10324b	stein.zip
dd1deac5a8f17c7715e886b2077e9764902be06f	Zeidler.rar
6e424fc2ed793429a60fba34e5362f195a0345f9	aguirre.zip
4edbcac2eb92883077cc6fbd84f48c3ad89f4cfc	ajtai.zip
57c5cf23a558dd271c936b92377d76b310c94ad2	boyett.rar
8fb769442b43475d270a4f81a61a26e0cc6ba495	linnane.zip
2cdd02064d95853181db54038e79ac3f10962366	smith.zip

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About Cigital, Inc.

Cigital helps commercial and government clients assure software quality and improve software development processes. Our Software Quality Management (SQM) solutions drive down the cost of deploying quality software and ensuring software reliability, security and performance. Cigital's expert Consultants measure software quality by combining proprietary methodologies, tools and knowledge to perform full-lifecycle testing via a risk management framework. The resulting metrics are used to drive application readiness decisions and identify the most cost-effective areas for software process improvement. Founded in 1992, Cigital (www.cigital.com) is headquartered in Northern Virginia with additional offices in Boston.

cigital

Digitally signed by Paco Hope
Reason: I attest to the accuracy and integrity of this document
Location: Cigital, Inc.
Date: 2009.01.29 14:34:58 -05'00'

EXHIBIT B

Lederer

WHY POKER IS A GAME OF SKILL

Introduction:
The Highest Stakes Game

The vast majority of players who visit public cardrooms, private clubs, or play online recognize that it is a game of skill. Poker rewards smart play. Losing players can learn to play better and win. The very best players can even make poker their livelihood.

But any game involving a deck of cards – any *honest* game – contains an element of chance. Poker players and defenders of the game have struggled mightily to explain poker's reliance on skill to non-players and those opposed to gambling. We struggle at our peril, however, because whether poker is a game of skill determines the passage of laws and regulations and, more important, their enforcement. The future of poker depends on our ability to explain in a clear, consistent fashion what we instinctively know: that poker is a game of skill.

In the absence of state laws specifically allowing or prohibiting poker, enforcers and courts look to the state's general gambling laws. These laws usually prohibit "games of chance." (They may also prohibit "lotteries," which tend to include all games of chance.) For example, Florida makes it a crime to "set up, promote or play at any game of chance by lot or with dice, cards, numbers, hazards or any other gambling device …." Fla. Stat. Ch. 849.11. In Delaware, a person is guilty of a crime who "keeps or exhibits a gaming table … at which cards, dice or any other game of chance is played …." 11 Del. Code § 1406. Connecticut defines "gambling" as "risking any … thing of value for gain contingent in whole or in part upon lot, chance or the operation of a gambling device but does not include: legal contests of skill, speed, strength or endurance in which awards are made only to entrants or the owners of entries." 11 Gen. Stat. Ct. § 53-278a.

These are typical of state laws that could concern poker. In fact, Senator Jon Kyl, co-sponsor of the UIGEA, admitted that if poker was proven to be a game of skill rather than a game of chance, it would be exempt from the law. He wrote, "If poker gambling enthusiasts truly believe it is a 'game of skill,' they can gain an 'exemption' by proving that to a court. Under most definitions of 'gambling' in state laws, games of skill are not 'gambling' even if there is an entry fee and a prize to be won."

The Test:
Whether Skill Predominates Over Chance

The legal test in most states is simple but ambiguous: a game is considered to be a skill game if skill predominates over chance in determining the outcome of the game. *See State ex el. Tyson v. Ted's Game Ent.*, 893 Wo.2d 355 (Ala. App. 2002) ("the issue is whether the nature of the game is such that the role of chance in determining the outcome is thwarted by the skill involved, or whether chance meaningfully alters the outcome and thereby predominates over skill"); *Harris v. Missouri Gaming Com'n*, 869 S.W.2d 58 (Mo. 1994) ("a game escapes the Constitutional bar against lotteries if skill is predominant"); *Contact, Inc. v. State*, 212 Neb. 584, 324 N.W.2d 804 (1982) ("a game of chance is one in which the winner is determined by mere luck and not skill; the predominant nature of the game, i.e., skill or chance, determines its classification"); In re *Allen*, 27 Cal. Rptr. 168, 59 Cal.2d 5, 377 P.2d 280 (1962) ("It is the character of the game rather than a particular player's skill or lack of it that determines whether the game is one of chance or skill. The test is not whether the game contains an element of chance or an element of skill but which of them is the dominating factor in determining the result of the game.") *cf. Horner v. U.S.*, 147 U.S. 449 (1893) ("where skill does not destroy the dominant effect of chance, the scheme is a lottery").

The question of whether a game is predominantly a matter of skill or chance is a question of fact, not a question of law. *See, e.g., Bell Gardens Bicycle Club v. Dep't of Justice*, 36 Cal. App. 4th 717, 42 Cal. Rptr. 130 (1995). In *Bicycle Club*, the defendants, operators of California poker rooms, argued for the legality of awarding jackpots to losers of certain lowball hands. The Court of Appeal followed the California Supreme Court in *Allen*, stating "[I]t is the character of the game as revealed by its rules that the courts look to in determining whether the elements of skill or chance predominate in determining the winner." Even though the parties agreed that poker was a game of skill, the court rejected the argument that because lowball was a game of skill, lowball with a jackpot was also therefore a game of skill. The court listened to testimony from experts by both sides specific to how the jackpot operated and how it changed the character of the game, concluding that the jackpot aspect – not lowball poker with or without the jackpot feature – was not predominantly a test of skill.

This means every time a case is brought, the judge or jury will have to consider the facts and the record of that case to make the determination. Past cases have no precedential value, which is important because some of the cases deciding whether poker was a game of skill did not consider the facts developed in this paper. (Happily, we have the upper hand in this controversy. While past adverse decisions don't foreclose us from bringing our winning arguments, it is likely once we score some victories – using our reasoning, our facts, our data, and our experts – the legal challenges to poker as a game of skill will cease. Lawyers don't like to lose, and politicians don't like to back losing causes, especially as eighty million or more American poker players increasingly make their voices heard.)

Defining Terms:
What's Predominant?
What's Skill?
What's Chance?
What's a Game?
What's an Outcome?
What Determines an Outcome?

Nearly all the terms in this test are subject to interpretation and need to be defined.

Predominant – If some elements of poker are "skill" and others are "chance," the term "predominant" requires evaluating the relative amounts of skill and chance. Essentially, "predominant" means more than 50%. Courts use dictionary definitions of the term. The Massachusetts Court of Appeals recently, in In re *May's Case*, 67 Mass. App. 209, 852 N.E.2d 1120 (2006), offered the following definitions of predominant: "In *Webster's Third New International Dictionary of the English Language Unabridged* 1786 (1993), 'predominant' is defined as 'having superior strength, influence, authority, or position' .... Similarly, in the *American Heritage Dictionary of the English Language* 1427 (3d ed. 1996), 'predominant' is defined as 'having greatest ascendancy, importance, influence, authority, or force' ...." The court, in a footnote, added the following definitions: "In the *Oxford Pocket American Dictionary of Current English* 620 (2002), 'predominant' is defined as 'being the strongest or main element' .... *See also Black's Law Dictionary* 1177 (6th ed. 1990), where 'predominant' is defined as '[s]omething greater or superior in power and influence to others with which it is connected or compared'...."

In other contexts, courts use fifty-percent-plus as the standard for predominance. *See, e.g.*, *HCPI Ind. LLC v. Hamilton Cty. Prop. Tax Assessment Bd. of Appeals*, 867 N.E.2d 713 (Ind. Tax. Ct. 2007) ("HCPI did not present any evidence showing that [the facility] was used more than 50% of the time (*i.e.*, predominantly) to educate medical students or to provide charitable health care ...."); *Concentric Network Corp. v. Commonwealth*, 877 A.2d 542, 549-50 (Pa. Commw. Ct. 2005) (concluding that telecommunications carriers and cable networks have the "burden of proving the lines are predominately used (more than 50%) directly in the exempt activity in order to qualify for the tax exclusion.").

Because there are just two elements under consideration – skill and chance – the one that is more than 50% responsible in determining the outcomes is the predominant element.

Game – What constitutes a game of poker? Instinctively, most players would define a game as one session or one tournament. But because a player, no matter how skilled, can't win every tournament or every session, players define success over a far longer period – months, years, a career. Moderate advantages over short periods of time repeated night after night will add up to an unbeatable edge. Remember, casinos have an edge of just 1-2% on each bet, yet over the course of weeks, months, and years, their profits are huge and guaranteed, able to support some of the world's largest businesses and build its largest buildings.

Focusing on the long run, however reasonable it may be, in arguing whether skill or chance is predominant, is a trap. Opponents note that the best player in the world can lose a hand or two – or even an entire session – to a clearly inferior player, suggesting that if such a thing is possible, chance predominates over skill.

A few recent courts that ruled poker was not a game of skill measured the game as one hand. "[I]n poker, a skilled player may give himself a statistical advantage but is always subject to defeat at the turn of a card, an instrumentality beyond his control." *Joker Club LLC v. Hardin*, 643 S.E.2d 626 (N.C. App. 2007); *see also Garrett v. State*, No. CR-05-1058 (Ala. Crim. App. 2007) ("'Texas Hold'em' poker is fundamentally a game of chance, in that the outcome of the game ultimately depends on a random draw of the cards.") (concurring opinion).

Therefore, even though we should not concede that a poker game consists of a single hand, we should gear our arguments to the elements of skill and chance that exist in every hand of poker. Even under this restrictive and somewhat unreasonable definition, the skill element of poker predominates over the chance element. Of course, we are not referring literally to one hand. But if we look at a large number of single-hand events and compile the data, we can see how the skill element accounts for the outcomes of most hands.

Skill – Definitions of "skill," according to *Merriam-Webster's Collegiate Dictionary* (11th ed. 2005), include "the ability to use one's knowledge effectively and readily in execution and performance" and "a learned power of doing something." Courts and administrative agencies have had the opportunity to define "skill" in various contexts and it always involves the same concepts: learning, experience, improvement, judgment.

For instance, in disability cases involving the Social Security Administration, the skill level of the person applying for disability payments is frequently at issue. The SSA has defined "skill" as "knowledge of a work activity which requires the exercise of significant judgment that goes beyond the carrying out of simple job duties and is acquired through performance of an occupation which is above the unskilled level." *Fines v. Apfel*, 149 F.3d 893, 896 (8th Cir. 1998) (quoting Social Security Ruling 82-41). Another federal court earlier quoted another part of an SSA ruling describing a skill as "practical and familiar knowledge of the principles and processes of an art, science, or trade, combined with the ability to apply them in practice in a proper and approved manner." *Tom v. Heckler*, 779 F.2d 1250 (7th Cir. 1985).

Under federal sentencing guidelines, judges are supposed to give harsher sentences to defendants who use a "special skill" in committing or concealing a crime. U.S. Sentencing Guidelines § 3B1.3. The comments to the Guidelines define a special skill as "a skill not possessed by members of the general public and usually requiring substantial education, training or licensing." An important element of skill in this context is that it be *acquired.* According to the U.S. Court of Appeals for the First Circuit, "We have held that a defendant need not necessarily have formal education or training in order to be found to possess a special skill. A special skill may also be acquired through experience or self-tutelage." *U.S. v. Prochner*, 417 F.3d 54 (1st Cir. 2005).

The *skill* element in poker is the betting. (Betting is defined here in its broadest sense, including folding, checking calling, raising, and, in games where the limits are not fixed, the amount bet.) Though there are arguably many different skills necessary to be a good poker player, like patience, ability to read one's opponents, aggression, courage, and emotional stability, all of those skills are expressed through the betting. For example, a player may detect a tell in his opponent that means he is bluffing. Though the player's skill at reading leads to knowledge of the bluff, it will be the great call he makes that will actually win the pot. If you were to, later, look at that particular hand history, there would be no record of the read, only a call with a weaker than normal hand that won the hand.

The skill element of betting is determined solely by free will and the players' own actions. "The word 'skill' speaks to the ability, through the application of human physical or mental capacity, to actually cause a desired outcome of a game when the game is played." *State* ex rel *Tyson v. Ted's Game Enterprises*, 893 So.2d 355 (Ala. Civ. App. 2002). "Skill increases the probability of 'winning.' In skill games, one person can be a better player than others." *Harris v. Missouri Gaming Com'n*, 869 S.W.2d 58 (Mo. 1994).

The betting element of poker contains all the elements associated with skill. Players can learn about betting, either through experience or by instruction. Certain players do better over statistically significant periods of time, primarily through increasing the amount they win on their winning hands and reducing the amount they lose on losing hands. Losing players, with experience, can and do become winning players, through the same means.

For years, poker players have made the fundamental mistake of confusing "skill" with "advantage." We know that certain poker players apply the skill of the game better than others. This leads to an advantage, which leads to profit over reasonable periods of time. But how does one's advantage determine the outcome of a poker hand? It is hard to say, and as long as we make this error, we will be doomed to losing this argument in court. In fact, the very nature of how you lose when you have an advantage – chance, the luck of the draw – proves our opponents' argument. An Alabama court expressed the oft-stated belief that "A player may be 'skilled' at 'playing the odds,' but he is still 'playing the odds.'" *State* ex rel. *Tyson v. Ted's Game Enterprises*, 893 So.2d 355 (Ala. Civ. App. 2002).

Furthermore, the fact that poker is a game of skill should not rest on whether or not the participants in a particular game are good players. To better understand this, let's look at a game like golf that everyone can agree is a game of skill. Golf itself does not change its character merely because of the level of skill of the particular players. When a golfer hits his drive 50 yards into the water, his skill (or lack of skill) did that. There is chance in golf (*e.g.*, imperfections on the green, gusts of wind), but this particular swing was all about a lack of skill leading to a bad result. The skill elements in golf are the elements that are completely in the control of the players: swing, club selection, aim. Skill elements don't have to be applied skillfully to be skill elements. They simply must be the part of any game to which the players try to apply their skill. It would be absurd to say that Tiger Woods is playing a skill game but a duffer playing at the country club is not. Tiger Woods is simply better at applying the skill elements of the game than the duffer.

Watching how a skilled poker player operates demonstrates how mastery of the skill element leads to success, but the choice to play the game skillfully does not make the game any less a contest of skill. "It is the character of the game rather than a particular player's skill or lack of it that determines whether the game is one of chance or skill." In re *Allen*, 277 Cal. Rptr. 168, 59 Cal.2d 5, 377 P.2d 280 (1962).

Chance – The California Court of Appeal has approved of the following jury instruction defining skill and chance:

> Skill is defined as the knowledge or the means or methods of accomplishing a task; the ability to use one's knowledge effectively and readily in execution or performance; dexterity or coordination in the execution of learned physical or mental tasks; a learned power of doing a thing competently; a developed or acquired aptitude or ability; a coordinated set of actions which become smooth and integrated through practice. Chance is defined as something that happens unpredictably without any discernable human intention or direction and in dissociation from any observable pattern.

People v. Shira, 62 Cal. App. 3d 442, 133 Cal. Rptr. 94 n. 12 (1976) (quoting from *Webster's Third New International Dictionary*).

The *chance* element in poker is the random distribution of the cards due to the shuffle. This element determines which cards each player receives and in what order they receive them. In community-card games, this includes the make-up of the community cards and the order in which they appear. Chance, by definition, is evenly distributed over time to all participants in a game and can't be predicted based on past results. If the cards were the sole determiner of poker outcomes, poker results would be evenly distributed (completely random) among all poker players. And a player's past performance would have no predictive value on future results.

Outcome – Defining the outcome of a hand of poker (or the more amorphously defined "game" of poker) as simply as who wins or loses is incomplete. In fact, defining the outcome of a hand as who wins the pot tilts the outcome strongly in favor of someone who wants to ascribe the outcome entirely to the luck of the draw. If betting is the skill element in poker, then the decision to bet or not bet (or, in certain games, how much to bet) has an effect not just on who wins the hand but on how much is won. A player who folds a losing hand earlier, at a smaller loss – or even at no loss – has exercised skill to minimize his loss.

If poker was a game where the result of the hand was simply who won it and all hands were equal, the player who won the most hands would be the most successful. Actually, the opposite is true. The better a player becomes at poker, the more selective that player becomes in choosing which hands to play. Winning poker players tend to play fewer hands, but they win a greater percentage of the hands in which they make a significant investment. More skilled players also win bigger pots when they win a hand, because they exercise skill in their betting decisions.

However, you will see we can still prove that skill predominates over chance in poker allowing for the narrowest definition of "outcome"— simply, the winner of the pot.

Determining – Nearly all statutes that define the test of whether a game is one of skill or chance involve the word "determine." How do you figure out what "determine" means when used in phrases like "determines the outcome" or "the winner is determined"?

The word "determine" is a common one and courts generally use its common-sense meaning. *See Mayalli v. Peabody Coal Co.*, 7 F.3d 570 (7th Cir. 1993). According to *Merriam-Webster's Collegiate Dictionary* (11th ed. 2005), the applicable definition of determine is "to fix the form, position, or character beforehand: ORDAIN <two points determine a straight line>; to bring about as a result <demand determines the price>." The definition of determine in the *Random House Dictionary of the English Language* (2d ed. 1987) includes "to cause, affect, or control; fix or decide causally." Similarly, *The American Heritage Dictionary of the English Language* (4th ed. 2006) defines this usage of determine as "to be the cause of, regulate."

In employment discrimination cases, where the "determining factor" of an employee's termination is usually an issue, the courts have long considered "determine" as synonymous with "but for" or "causative" or "deciding". *See, e.g., McDonald v. Santa Fe Transportation Co.*, 427 U.S. 272, 282 n.10 (1979) ("no more is required to be shown than that race was a 'but for' cause"); *Loeb v. Textron, Inc.*, 600 F.2d 1003 (1st Cir. 1979) ("determining factor" means "but for his employers' motive to discriminate against him because of his age, he would not have been discharged"); *Pannell v. Food Services of America*, 61 Wash. App. 418, 810 P.2d 952 (1991) (approving the employers' argument that "'determining factor' is a term with a commonly understood dictionary meaning of 'causal' or 'deciding'").

How Skill Predominates Over Chance in Poker:
No-Show Hands

There are two ways you can win a hand of poker: you can have the best hand of the remaining hands at a showdown, or you can be the last player in the hand after everyone else has folded. It is worth examining showdown hands and no-show hands separately, to understand the roles of skill and chance in each. Although skill predominates over chance in showdown hands, the chance element in a showdown has drawn so much attention that it has obscured two overwhelmingly important facts: (1) most hands are not shown down; and (2) in no-show hands, the cause of the outcome is skill and chance plays practically no role.

Too often, non-players look at poker and see a random distribution of cards and a big-money showdown, where the player with pocket aces was lucky because he had pocket aces and his opponent was unlucky enough to have pocket kings. Or he was unlucky enough to have pocket aces and his opponent with pocket kings caught a lucky king on the river. Courts fall back on clichés like "A player may be 'skilled' at 'playing the odds,' but he is still 'playing the odds.'" *State* ex rel. *Tyson v. Ted's Game Enterprises*, 893 So.2d 355 (Ala. Civ. App. 2002). In the recent *Joker Club* case in North Carolina, the state's sole witness testified "he had seen a television poker tournament in which a hand with a 91% chance to win lost to a hand with only a 9% chance to win." *Joker Club LLC v. Hardin*, 643 S.E.2d 626 (N.C. App. 2007) (*i.e.*, somebody hit their inside straight draw on the river.) Even poker players and defenders of the game fall into this trap, acknowledging that chance can determine the result over the "short term" but skill predominates over the "long term."

This is completely false. Even over the short term – a single hand of poker – the skill element, the free will of the participants expressed by their betting decisions, determines the outcome far more often than chance possibly could. Looking at millions of hands, encompassing Limit and No-Limit Hold 'Em, tournaments and cash games, low limits and high limits, and players at all levels of skill, it turns out that in only 18% of the hands is there a showdown where the cards are revealed (all of the percentages quoted here are taken from a random sample of 16 million hands of nine-handed hold 'em taken from a leading online poker site; these hands include no-limit, pot-limit, limit, and tournament hands in all of these variations. All hands were counted equally, with no bias towards larger or smaller games. In fact, over 60% of the hands counted were played at micro-limits. This seems to dispel the myth that though high-limit players are playing skillfully and folding a lot, amateurs don't).

The showdown is the anomaly; 82% of the time the betting alone determines the outcome. In these no-showdown instances, the chance element, the random distribution of the cards, had no role in determining the outcome. According to the rules of poker, when there is only one player left and no one will call that player's bet, that player wins the pot. The cards still to come have no role. In fact, the cards the players have received have no role. Whether the best hand (defined as either the best hand to that point or the best hand after all the cards are dealt out) wins a no-showdown hand is both incidental and irrelevant to the outcome.

Can the cards influence the betting? Certainly. But it is the betting itself that actually *determines* the outcome. The players choose how they want to bet, and they may or may not take their cards into account. In fact, players often intentionally ignore the value of their cards when deciding to bluff at a pot. Likewise, all the other players in the hand who eventually fold may consider the value of their cards to varying degrees. The *cause* of one player winning a no-showdown pot was not the cards he or any of his opponents held. It was his action in betting and their actions in folding.

In this respect, what determines the outcome of a no-show poker hand is similar to what determines the outcome of a hole of golf. What determines the winner of a golf hole is which player put the ball in the hole in the fewest strokes. Changes in the wind, bounces and lies, spike marks on the green – they play a role but the winner is determined by who put the ball in the hole first. Golf is a game of skill regardless of the influence of chance on the way to the goal. Poker hands, likewise, when there is no showdown, are resolved through the actions of the participants. The outcomes of 82% of poker hands are not *determined*, as that word is defined, by the random distribution of the cards.

The beauty of poker is any hand can win under the right circumstances. This is a clear distinction from other casino games or betting on sports. In those other forms of gambling, a definitive result is always produced (*e.g.*, the banker won, the underdog covered the spread, the ball landed on black) and the winner is determined by who correctly predicted that outcome. In 82% of poker hands, that is not the case. There is no definitive outcome outside the control of the players. Their betting actions alone cause the result.

How Skill Predominates Over Chance in Poker:
Showdown Hands

In the 18% of poker hands where the cards are consulted at showdown and the chance element has an opportunity to significantly influence the outcome, skill still predominates. The betting determines the outcome more often, and to a greater degree, than the random distribution of the cards even when there is a showdown. The overwhelming effect of the players' betting actions on showdowns manifests itself in two ways: (1) getting the "luckiest hand" to fold before the showdown; and (2) controlling the amounts won and lost.

Getting the "luckiest hand" to fold; Texas Hold 'Em v. Luck Hold 'Em – If there was no skill element in poker, the outcome would be set as soon as the dealer completed the shuffle. The hole cards and community cards would already be ordained and it would simply be a matter of moving forward to the showdown to determine the winner. Even opponents of poker recognize that there is some free will exercised by the players – the betting– between the conclusion of the shuffle and the showdown. But, how much does the betting affect the determination of the winner in these showdown hands?

By constructing a game called "Luck Hold 'Em," we can demonstrate the difference between Texas Hold 'Em, with its elements of skill and chance in showdown hands, and a game *like* Texas Hold 'Em but with all the skill elements removed.

Luck Hold 'Em is played very much like casino games such as Baccarat or Casino War or Roulette. Nine players sit down at the table and bet an equal amount, say $1. Once the betting is complete, the dealer will deal each player a two-card hold 'em hand. It doesn't matter whether these cards are dealt up or down as there will be no skill element applied to determining the winner. The dealer will, like in Texas Hold 'Em, produce a three-card flop, followed by a fourth card and a fifth card. After dealing all the cards, the dealer will look at all nine hands and determine, using standard poker hand rankings, who wins the pot. In any single hand of Luck Hold 'Em, the players receive the same hole cards and community cards they would have received if the hand had been played as a normal hand of Texas Hold 'Em, only there is no betting.

Clearly, Luck Hold 'Em is a game of pure chance. The only thing that determines the outcome is the cards. And there can be no skill elements applied to influencing either the cards or the outcomes they produce – no checking, calling, raising, or folding.

In Luck Hold 'Em, the best hand, what we will call the "luckiest hand" after all the cards are dealt, wins 100% of the time. How similar are the results of Luck Hold 'Em hands to regular Texas Hold 'Em hands? Hardly similar at all. The data over millions of hands show that the "luckiest hand" wins a showdown hand in Texas Hold 'Em only 7% of the time. The other 93% of the time, the hand that would have won in Luck Hold 'em folded before the showdown or all the other players folded, awarding the "luckiest hand" the pot without a showdown. This means that in 60% of showdown hands (remember, we only have a showdown hand 18% of the time), the "luckiest hand" folded, allowing another player to win. The only explanation is that the intervening skill elements (the betting) were responsible for the drastic change in the results. If the "luckiest hand" had not folded, he would have won. Therefore, it can be argued it was his fold that determined a different winner—the outcome.

For example, chance would determine that a player dealt 8-4 would win a particular Luck Hold 'Em hand, because with community cards of 5-J-A-7-6, that player would end up with a straight. In the same hand, the player with A-A would be destined to lose. That would be the result in Luck Hold 'Em but it is an extremely unlikely outcome in Texas Hold 'Em. The players, using their skill in making betting decisions, would end up with an outcome where the A-A stays in and wins, while the 8-4 would fold.

That's not to say 8-4 doesn't *ever* beat A-A. 7% of the time the "luckiest hand" wins showdown hands, and a tiny number of those could be 8-4 beating A-A. Even in this circumstance, however, it is hard to ignore the element of skill. Maybe the player with 8-4 was so enamored with the possibilities of chance that he kept calling bets hoping fate was somehow on his side. But it seems more likely that the player with A-A made a conscious decision to not bet his strong hand and allowed the player with 8-4 to make that lucky hand.

At most, chance predominates in 7% of Texas Hold 'Em hands. These would be the hands where there is a showdown *and* the "luckiest hand" wins. It could be argued, however, that even in these hands, *but for* (remember the definition of "determine") the player who was dealt the "luckiest hand" using his free will to stay in the hand, he would not have won.

Skill and the Amount Won or Lost; the Neglected Skill of the Fold – As described in the definitions section, the "outcome" of a hand of poker should include not just who won the hand but also how much was won by the winner and how much was lost by each of the losers of the hand. Each hand of poker is not equal. If the players agreed to give some prize to the player who won the most pots at the end of a session, then winning pots would be important and, assuming the prize was substantial, the size of the pots would be secondary. But that's not the case in tournaments or cash games. There is a big difference between winning the blinds and antes by betting and having everyone fold vs. winning a big multiplayer pot. Likewise, the players who don't win the pot can, over the course of a hand, lose between zero and everything they have in front of them.

The betting actions of the players determine the size of the pot. Even if chance plays a role in who wins a showdown, the players' betting decisions alone determine its size.

The size of the pot is actually more important than the identity of the winner or loser. Strategies like slow-playing, check-raising, bluffing, and semi-bluffing are all designed to profoundly influence the size of the pot by manipulating the betting decisions of opponents. The data collected from millions of hands show that the most skillful players lose more pots but win a higher percentage of the pots in which they make a significant investment. Since a skilled player tends to lose little to nothing on the pots he loses, this strategy is very effective.

The most important thing players can do to reduce the amount they lose in losing hands is to fold early. In a nine-player game with a two-player showdown, think about what happened with the other seven players. Their decisions to fold were acts of free will in an attempt to play skillfully. The moment they folded their hands, they removed the element of chance from having any impact on their fortunes. Each of those seven players didn't win the hand. But more important, they either broke even because they folded before contributing to the pot, lost relatively nominal blinds or antes, or lost a substantial amount depending on how and when they folded. And, the timing of their fold was completely under their own control.

Overall, the impact of folding has been ignored by opponents of poker in their focus on the players in the showdown. There is no question that the decision to fold is an exercise of skill and not chance. But what impact does it have on the outcome of a hand of poker? Is it the null event poker's opponents presume by ignoring it?

The data show that the decision to fold may be the most important skill in poker. The results of millions of hands by thousands of players at all levels consistently show that the players who fold the most are among the most profitable. So even if, in a particular hand, chance played a significant role in determining who of the remaining two players won a pot at showdown, the betting actions of those two players along with the equally important folding actions of the other seven players were predominant in determining the size of the pot and, therefore, the outcome of the hand.

And this is not an elusive skill that can be mastered only through years of experience or by knowledge hoarded by top professionals. To the contrary, learning to fold is almost universally regarded as the first skill new poker players develop. In fact, the strongest relationship between most hands folded and profitability is typically found in the lowest-stakes games.

Long-Term Results

Prove That Betting is a Skill -- And It Matters!

Poker players engaging in the argument over whether poker is predominantly a game of skill or chance fall into a trap by saying, "It's a game of skill over the long term but luck plays a major role in the short run." As previously described, this is a dangerous and inaccurate concession. Consequently, the analysis to this point has ignored "the long term." Long term results, however, confirm that the skill element of betting is a skill that can be used to create good outcomes over time.

If the skillful use of betting did not lead to favorable outcomes, poker results, over the long term, would be evenly distributed. In poker, since the cards are subject the shuffle (either physical or virtual), they can be shown to be a chance element in the game. Since some chance does exist in poker, unless there was a skill element in the game, we would see the long term results of poker players distribute evenly over time, just like the cards. A player's history would have no value in predicting their future results.

But poker outcomes are not evenly distributed. There are people (too many to be explained by mere chance) who derive regular income from their results at poker. The federal government has long recognized that poker players are engaged in a "trade or business" and, for tax purposes, their income is "earned income." *See Baxter v. U.S.*, 633 F. Supp. 912 (D. Nev. 1986). In *Baxter*, the court noted that "Plaintiff William Baxter is a professional gambler. He has been gambling since age 14 and has never had any other occupation. During the years at issue, poker was Baxter's primary gambling activity." In determining that Baxter's income was "earned income" and not like a lottery payout, the court concluded "Clearly, Baxter's $1.2 million gaming income for the years at issue was not derived from his passive investment of capital in a series of risky ventures. Baxter expended substantial time and energy playing poker.

Baxter consistently won at poker because he possesses extraordinary poker skills. Any argument that Baxter's gaming income is not based on his personal expenditure of time, energy, and skill is meritless."

Courts should take judicial notice, or proof could easily be furnished, to confirm this. There are men and women – their names are known where they play and sometimes throughout poker – who play regularly in cash games who have won money every year for a decade or more. Most of them have rarely had a losing month. In tournament poker, even though 90% of the field typically receives nothing for playing and most of the prize pool is distributed among just the top few finishers, there are players who have consistently shown a profit for more than a decade.

Not only do we see incontrovertible evidence that winning players tend to continue winning, but losing players tend to lose less over time, and some even become winners (see the Case Western of skill in poker for recent proof of this). This suggests that there is something being learned at the poker table. The core element of skill is that it can be learned. But what could this skill be? The only logical conclusion is the betting. The data from millions of hands confirm it: results are not random, winners tend to win, some losers learn to win, and the skill at betting (expressed not by simply winning more hands, but by increasing profits on winning hands and reducing losses on losing hands) is the reason. Though a poker pro may have hundreds of skills, all of those skills are ultimately expressed through their betting. And their ability to make betting decisions more skillfully than their opponents is the reason for their success.

The existence of the poker publishing industry also demonstrates that poker has a huge skill component. Mason Malmuth's Two Plus Two Publishing has sold over a million copies of books on poker strategy. David Sklansky, who has written several books for Two Plus Two, has had more books at one time in the Top 100 on Amazon.com than any other author except J.K.

Rowling. Doyle Brunson's pair of *SuperSystem* books have each sold over a hundred thousand copies. Dan Harrington's *Harrington on Hold* 'Em trilogy has also sold hundreds of thousands of copies. Granted, there are books on how to win the lottery, but how many of them have sold hundreds of thousands of copies? How many have sold copies year after year for decades? How many authors of those books are themselves consistent winners, as Doyle Brunson, Dan Harrington, David Sklansky, and Mason Malmuth are at poker? How many authors of "roulette strategy" books are sufficiently in demand to write successful follow-up books?

Furthermore, the content of these books focuses on how to improve at the skill element, the betting. Success and failure in poker are all about the sum of a player's betting decisions. Dan Harrington emphasizes in the introduction of his first book that "The no-limit form of hold 'em poker is very advantageous to good players for a simple reason. By making superior deductions about the hands their opponents hold, *they can make bets that offer their opponents more chances to make errors*." D. Harrington, 1 *Harrington on Hold 'Em* 14 (2004) (emphasis in original).

David Sklansky's *The Theory of Poker*, one of the best-selling strategy books on any game, emphasizes in an early chapter that the goal is to win big pots. Betting is required to win those pots without a showdown and to improve a player's chance of winning a showdown. "When the pot is big, you want to win it right away. To try to win it right away, you should bet and raise as much as possible, hoping to drive everybody out, but at least reducing the opposition." D. Sklansky, *The Theory of Poker* 77 (4^{th} ed. 2002).

In fact, if you glance through Sklansky's book, the main strategies all concern methods of betting. Betting strategy, he repeatedly explains, defeats luck by getting the "luckiest hand" to fold. Knowing the right situations in which to bet also allows players to win with inferior hands (*i.e.*, bluffing) and, by skill, improves their ability to "get lucky" by keeping opponents from betting them off their drawing hands and increasing the size of the win when their draw hits. (*I.e.*, semi-bluffing, betting to get free cards.) Simply looking at the headings under "Chapter Thirteen – Raising" shows how the betting decisions are the skill element and how the betting dominates the outcomes:

- Raising to Get More Money in the Pot
- Getting More Money in the Pot by Not Raising
- Raising to Drive Out Opponents
- Raising as a Means of Cutting Down Opponents' Odds
- Raising to Bluff or Semi-Bluff
- Raising to Get a Free Card
- Raising to Gain Information
- Raising to Drive out Worse Hands When Your Own May be Second-Best
- Raising to Drive out Better Hands When a Come Hand Bets
- Raising Versus Folding or Calling

The people reading the millions of copies of poker strategy books written and edited by winning players have themselves become better players. It is possible to get better at poker, which could not happen if chance was the only significant element. Has anyone ever scientifically proven that it possible to get better at keno or baccarat or roulette? The data show that poker players improve with experience and losing players can become winning players. Furthermore, the evidence ties this to the betting. Players win more as they become more skilled at folding and they win more as they win more big pots. The number of pots won – the emphasis of opponents of poker, especially the number of showdowns won – has almost no bearing on success or failure at poker; in fact it is a contra-indicator of success.

There are a number of studies that show that making correct decisions in poker will lead to insurmountable edges over time. Poker is not a game where the better player wins every hand or every night. But, the best players win almost every month and will certainly win every year. It takes patience, guts, psychology, and a keen understanding of math to succeed at the game.

Poker is a true meritocracy. Young and old, male and female, white and black -- they all have an equal chance to win. The player does not have to try to overcome an unbeatable house edge to win from a huge publicly-traded corporation as when they play casino games. We all know that almost never happens.

Instead, all they have to do is use their skill better than the players they are playing against. What could be fairer?

EXHIBIT C

Chance and strategy in Poker

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Summary

The aim of this analysis is to quantify the impact of chance versus strategy in the game of *Texas Hold'em Poker*. It thereby complements N. Alon's [1] work on this subject by broadening the game model considered. The results were obtained with a theoretical study carried out by digital simulations of virtual poker games. We concluded that, for a sufficiently high number of consecutive games, it is clear that strategy rather than chance is the overiding factor in the outcome of a *Texas Hold'em Poker* game.

Key words: *Texas Hold'hem Poker*, chance, strategy, Monte Carlo simulations.

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1 Introduction

Poker being a game increasingly practised, a question with important legal consequences has come to the fore over the last few years: *Is poker a game where strategy prevails over chance?* This study endeavours to give an answer which is mathematically rigourous to the question detailed from a legal point of view in [7].

Of course, a very large number of studies suggest various poker strategies (see [3] for example), but very few deal with the question of chance in games results. They are mainly focused on game theory problematics searching for strategic balance between players, or artificial intelligence allowing a progressive adaptation to opponents' behaviour. The only study which seems to answer this question satisfactorily is N. Alon's [1]. Studying a simplified version of a game of poker, his analysis concludes that Poker is essentially a game of strategy. Indeed, thanks to the Central Limit Theorem, a powerful tool in the theory of probability, he shows that the strategy employed is a determining factor in the result of a sufficiently large number of games.

In this study, we first validate N. Alon's findings and we then generalise them. Indeed, N. Alon considers a simplified version of the game of poker, taking into account only the last round of the game where all the community cards are known. He studies mainly games between two players: Alice who has a well-defined strategy and Bob who plays in a random manner. He assumes that Alice has an intimate knowledge of the way Bob plays which gives her a considerable advantage. We put ourselves in a real game of *Texas Hold'em poker* model, in following the pattern of the various stages of the game: Preflop, Flop, Turn, River. In this more general model, even if we only consider two players, the analytical calculations carried out by N. Alon can not be done so we have used digital simulations. In other words we have performed a digital simulation of virtual poker games between Bob, who plays in a random manner, and Alice who follows a well thought-out strategy, and we have analysed the results.

In this model which is very close to reality, we draw conclusions that are very similar to N. Alon's: **for a sufficiently high number of games, the strategy employed is a determining factor in the outcome of a game of** *Texas Hold'em Poker.*

This study is organised as follows: first, we present the rules and we describe the course of a *Texas Hold'em Poker* game. Then, we study N. Alon's results and we suggest a more general game model, for which we specify interesting strategies. Finally, with the assistance of virtual games digital simulations, we estimate the probability of winning using these strategies. The strategies selected are not necessarily the best ones in the end, but they have the advantage of defining simple and realistic decision criteria for a poker player who is able to assess his own skills. These criteria can also be simply adapted to multi-player games. As we will see, the strategies are sufficient to ensure very high probabilities of winning.

2 The game of Texas Hold'em Poker

We will attempt in this study to consider a type of poker game as close as possible to that of the *Texas Hold'em no limit* as it is officially described in [6] and the principles of which are laid out in this section.

2.1 The game rounds

Texas Hold'em Poker is played with a standard 52-card deck and each game is punctuated by an alternation of card dealing and rounds of betting. There are 4 of these phases and card dealing takes place in the following order.

Preflop: Each player is dealt 2 pocket cards face down.

Flop: Three community cards (the flop) are now dealt face up.

Turn : A fourth community card (the turn) is revealed

River : A fifth community card (the river) is revealed

At the end of each phase of card dealing, a betting round starts. Players place their bets one after the other and if a player wishes to stay in the game, he or she must at least match the biggest stake. This round of betting stops when all the players who are still in the game have bet the same number of chips. All the bets make up what is called **the pot**. Finally two scenarios are possible: either there is only one player left in the game and he or she wins the pot or there are several players left and the game moves to the next stage.

If, after the last round of betting following the river, several players are still in the game, they are rewarded according to the value of their hand. Each player can then make use of his or her 2 pocket cards and the 5 community cards to make the best 5-card hand possible out of the 7 available. The player with the best hand then wins the pot and, in the event of two or more hands being worth the same, the players concerned split the pot. The various possible poker hands are described in the following section.

A game of poker is made up of a succession of rounds of this type, rounds where players take it in turns to bet first. At the beginning players have the same amount of chips available to them and they fold when they have no chips left. In order to encourage players to bet, compulsory stakes are added at the first round of betting (**blind** or ante) for certain players on the table.

2.2 Hand ranking

At the end of a poker game, each player still in the game must reveal his or her cards and the strength of his or her cards is determined by the best 5-card hand he or she can assemble out of the 7 available to him or her. In a 52-card deck, there are 2,598,960 5-card unordered hands possible and all these combinations of cards are separated in ten categories according to their probability of appearing:

Royal Flush: Ace, King, Queen, Jack and Ten of the same suit.

Straight Flush: Any straight with all 5 cards of the same suit.

Four of a kind: 4 cards of the same rank.

Full house: 3 cards of the same rank together with any 2 cards of the same rank.

Flush: 5 cards of the same suit which are not consecutive.

Straight: 5 consecutive cards of different suits.

Three of a kind: 3 cards of the same rank.

Two-pair: 2 cards of the same rank together with another two cards of the same rank.

One-Pair: 2 cards of the same rank.

High card: Any hand that does not make up any of the above-mentioned hands.

The rarer a 5-card hand is the more it is worth. Within each category, hands are ranked according to how high the cards are. All 5-card hands can therefore be ranked amongst each other, with sometimes the possibility of a draw. Let us now consider combinations of 7 cards where the best 5-card hand out of the 7 possible is kept. There are then 133,784,560 possible combinations of 7 unordered cards. The following table shows [1] and [9], for each hand category, the number of 5 and 7-card possible combinations and their probability to occur.

	5-card combination		7-card combination	
	Number	Probability in %	Number	Probability in %
Royal Flush	4	1.5 10−4	4324	3.2 10−3
Straight Flush	36	1.4 10−3	37 260	2.8 10−2
Four of a kind	624	2.4 10−2	224 848	0.17
Full house	3 744	0.15	3 473 184	2.60
Flush	5 108	0.20	4 047 644	3.03
Straight	10 200	0.39	6 180 020	4.62
Three of a kind	54 912	2.11	6 461 620	4.83
Two-pair	123 552	4.75	31 433 400	23.5
One-Pair	1 098 240	42.3	58 627 800	43.8
High card	1 302 540	50.11	23 294 460	17.4

3 Analysis of N. Alon's article

3.1 Structure of the analysis

In his analysis [1], N. Alon considers a variant of poker game suggested by Von Neumann and Morgenstern [5]. He considers a game with two players, Alice and Bob which works as follows:

1. Each player is 'dealt' a random number drawn in even interval [0, 1]. This number represents the value of their cards and is represented with an x_A for Alice and x_B for Bob.
2. Both players then decide simultaneously to bet 1 chip or to fold.
3. If one of the two players has folded, the game stops and there is no exchange of money. If both players have bet, the player whose cards have the highest value wins the amount wagered by the other, i.e. 1 chip.

This simple game variant captures the essence of the last round of a poker game well, when all the community cards are revealed. Indeed, there are then $C_{47}^{2} = 1081$ possible combinations of two cards for each player. Disgarding the draws and superpositions between the various combinations, these combinations can be ordered. Each player is therefore dealt cards with a value ranging from 1 to 1081. This range, divided by 1081, resembles the values of the card combinations x_A and x_B considered in the variant. At the end of the game, if both players have bet, the player with the highest range wins the pot. The benefit of considering such a simplified game is that it allows us to perform analytical calculations in an explicit manner and to calculate the players expected winnings.

N. Alon mainly considers a two-player game where Bob plays in a random manner and Alice plays in a more strategic way by adapting to Bob's game. Bob's random behaviour can be represented with the following pattern: he bets 1 chip with a probability of $p := 1/2$ and folds therefore with a probability of $1 - p = 1/2$. Alice knows Bob's strategy and seeks to adapt her strategy to his behaviour.

3.2 The Results

N. Alon then demonstrates that the optimal strategy for Alice consists in betting if and only if the value of her cards is above 1/2. He then shows that her winning odds are 1/8 at each game and that the associated variance is 15/64. In other words, Alice's average winning odds at each game are 1/8 and the manner in which they fluctuate is characterised by a variance of 15/64.

Thanks to the Central Limit Theorem, he can then estimate the probability of Alice losing following a sufficiently high number n of consecutive games. The results are very convincing; after for example 350 games, he observes that the number of times Alice looses is less than 1 in a million. He thus naturally comes to the conclusion that for a sufficiently high number of games, strategy is a determining factor in the outcome of a game.

He also broadens these results by adding compulsory bets (blinds) in each game and he briefly analyses the instance of a game with more than 2 players.

3.3 The main limitations

The results presented in N. Alon's article are absolutely valid and pertinent but it is nevertheless regrettable that the analysis should be restricted to a simplified version of poker game which only takes the last round of the game into account. It is true that the analytical calculations proposed would be difficult to apply to the complete version of the game. We will therefore broaden N. Alon's findings to a real poker game model by replacing the analytical calculations with digital simulations.

In the same way, the fact that Alice knows Bob's strategy, in other words the probability p with which he bets one chip, is a problem. Indeed, it *is* rather surprising that she would be able to identify her opponent strategy that easily. Here, we will aim to construct reasonable strategies which seem adapted to Bob's various random ways of playing.

4 The model considered

We will therefore use a similar approach to that of N. Alon, but with a game very close to the real rules of *Texas Hold'em Poker*. In the first instance, in order to better understand the required strategies, we will limit ourselves to games with 2 players.

4.1 The game

We will consider that two players, Alice and Bob are sat around a table of poker.

4.1.1 Stages of the game

The game progresses in 4 rounds:

Preflop: Alice and Bob are dealt 2 cards each. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, the game continues.

Flop: the first three community cards are dealt face up. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, the game continues.

Turn: the fourth community card is revealed. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, the game continues.

River : the fifth community card is revealed. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, they compare hands. The player with the best hand wins the pot or in the event of a draw, the two players share it.

Note that the main differences between a real *Texas Hold'em* Poker game and this version are as follows:
- There is no compulsory bet or blind
- The amounts wagered at each round are fixed and equal to 1
- At each round the players decide simultaneously whether they want to bet
- It's a 2-player game

We will include in section 4.3 some variants of the game described here and they will take the following differences into account: blind, variable bets, several players.

4.1.2 The players

Just like in N. Alon's article, we consider that 2 people play against each other:
- Bob who has a random strategy with a probability p of betting and (1 -p) to fold at each round of the game.
- Alice who aims to adapt her strategy to Bob's. She suspects that Bob's strategy is random but she does not know the probability p which governs his decisions. She makes her decisions by estimating what his pocket cards might be.

4.2 Alice's optimal strategy

Because Alice does not know the probability p characterising Bob's behaviour, she will devise a strategy not dependent on p. She observes however that at the last round where the pot was not nil, the situation most favourable to Bob is if he never folds, i.e. p=1. To define her strategy, she therefore considers the case where Bob never folds. It is this strategy which she will use later even when p is different from 1.

4.2.1 The optimal strategy during the River

Alice's optimal strategy during the river is easy to determine. Let us assume that the pot is worth P, and that Alice has cards which mean that the probability that she will win is X. Knowing that Bob does not fold and discarding the possibilities of a draw, if Alice bets, her chance of winning is $X(P + 2) - 1$. Her chance of winning being nil if she folds, we can deduce that

$$\text{Alice must bet at the river if and only if} \qquad X \geq \frac{1}{P+2}$$

The interpretation of this boundary is clear and can be read in the following manner: as she wagers one chip in the hope of winning $P + 2$, it is in her interest to play if and only if her probability of winning is greater than $1/(P + 2)$. Note that the more there is in the pot, the least important it is for Alice to have good cards in order to bet.

To estimate her probability of winning X, all Alice has to do is count the number of hands she could beat amongst the $C_{45}^{2} = 990$ other possible hands. This calculation, easily performed by a computer is of course impossible for a human brain. However for experienced players, it is not difficult to estimate X relatively precisely. In order to adapt to the reality of a player who estimates his or her probability X of winning with possibly one error, we will present in section 5.2.4 the results of digital games where we have artificially added a random measuring error on the estimation of X.

4.2.2 The optimal strategy during the Turn

Let us assume that Alice uses in the last round the strategy previously described. Let us then work out what her strategy should be at the previous round. Note that P is the amount in the pot and X is the random variable equal to Alice's winning odds at the last round of the game. The variable X is random in the sense that it is not yet known; it can indeed have 46 different values depending on the last community card. If Alice decides to bet in this round, her odds of winning are:

$$\mathbb{E}\left[(X(P+4) - 1)1_{X \geq 1/(P+4)} - 1\right].$$

Accordingly, as she would win nothing by folding, it is in Alice's interest to bet if and only if

$$\mathbb{E}[X1_{X\geq 1/(P+4)}] \geq \frac{1 + \mathbb{P}[X \geq 1/(P+4)]}{P+4}$$

This time, Alice's strategy which when analysed appears more complex is in fact also very intuitive. The principle is the following: There is no point in Alice betting at this round if she does not bet at the following one. Therefore she hopes to win $P + 4$ by betting 1 at this round and 1 at the following. She is testing whether her odds of winning by betting at the last round are greater than the total she has wagered divided by her winnings. Calling R the event where Alice bets during the river,

Alice must bet during the Turn if and only if $\mathbb{E}[X1_R] \geq \dfrac{1 + \mathbb{P}[R]}{P+4}$

4.2.3 Optimal strategy during the Flop

The same type of rationale can be applied to the flop if a strategy has been defined thereafter. Calling T, the event where Alice bets during the turn and $T \cap R$, the event where Alice bets both during the turn and during the river, Alice's odds of winning pot P are the following:

$$\mathbb{E}[(X(P+6) - 1)1_{T\cap R} - 1_T - 1] .$$

One can see that Alice will have to bet if and only if her winning probability is greater than the ratio between the potential amount she has wagered and the total pot.

Alice must bet during the Flop if and only if $\mathbb{E}[X1_{T\cap R}] \geq \dfrac{1 + \mathbb{P}[T\cap R] + \mathbb{P}[R]}{P+6}$

4.2.4 Optimal strategy during the Preflop

Obviously, the same rationale always apply when none of the community cards are yet available. Calling F the event where: "Alice bets during the Flop" we know that

Alice must bet during the Preflop if and only if $\mathbb{E}[X1_{F\cap T\cap R}] \geq \dfrac{1 + \mathbb{P}[F\cap T\cap R] + \mathbb{P}[T\cap R] + \mathbb{P}[R]}{8}$

4.2.5 Optimal strategy when the stakes vary

So far we have assumed that the amount wagered at each round equalled 1 chip. Of course in reality, stakes vary. Taking into account the variability of these stakes renders the previous analysis much more complex. Indeed in this instance the criterion used whereby the odds of winning are maximised does not seem very appropriate. However the results previously obtained can be easily extended to the scenario where the stakes at each round are different but determine ,or vary according to, the pot. Empirically, in a poker game, the closer the river is the more the stakes go up.

Let us call m_P, m_F, m_T and m_R the stakes for each player respectively during the preflop, during the flop, during the turn and during the river. We deduce that the optimal criteria which Alice must consider at each round of the game for a pot of P are the following:

River: $X \geq \dfrac{m_R}{P + 2\, m_R}$

Turn: $\mathbb{E}[X1_R] \geq \dfrac{m_T + m_R\mathbb{P}[R]}{P + 2 * (m_T + m_R)}$

Flop: $\mathbb{E}\,[X1_{T\cap R}] \geq \dfrac{m_F + m_T\mathbb{P}[T\cap R] + m_R\mathbb{P}[R]}{P + 2 * (m_F + m_T + m_R)}$

Preflop: $\mathbb{E}\,[X1_{F\cap T\cap R}] \geq \dfrac{m_P + m_F\mathbb{P}[F\cap T\cap R] + m_T\mathbb{P}[T\cap R] + m_R\mathbb{P}[R]}{P + 2 * (m_P + m_F + m_T + m_R)}$

4.3 The strategies put into action

In order to make this analysis more realistic and more reader-friendly, we will simplify the criteria taken into account in Alice's decision making process. We will therefore not use the optimal strategy which Alice has available to her. Here we will consider only strategies that are merely based on Alice's hand odds of winning versus another hand. In other words, in each round, Alice knows $\mathbb{E}\,[X]$ where X represents her probability of winning at the last round of the game. Of course, $\mathbb{E}\,[X]$ is recalculated at each round of the game by taking into account the new community cards. For an experienced player, it is easy to know the potential of one's hand and to see how it evolves as the flop, the turn and the river unfold.

4.3.1 The reference strategy

We will consider strategies where, at each round of the game,**Alice bets if and only if $\mathbb{E}\,[X] > x(P)$** where P is the value of the pot and x a decision function. We still have to specify the functions x_P, x_F, x_T et x_R corresponding to each round of the game: preflop, flop, turn and river. To simplify the presentation, we will assume that the stakes at each round are equal to one chip.

First of all, during the river, X is no longer random as all the cards are known and the criteria of optimal strategy can be applied:

$$X \geq \frac{1}{P+2} \quad \text{namely} \quad x_R(P) := \frac{1}{P+2}$$

Let us now consider the other rounds of the game. The previous optimal criteria can not be applied but they give a good idea of the threshold function x_P, x_F et x_T that is reasonable to use. Let us assume that at each round of the game, Alice decides to bet thinking that she will not fold at any of the following rounds. Then her decision criteria during the river, the flop and the preflop become respectively

$$\mathbb{E}[X] \geq \frac{2}{P+4}, \quad \mathbb{E}[X] \geq \frac{3}{P+6} \quad \text{and} \quad \mathbb{E}[X] \geq \frac{4}{P+8}$$

The criteria therefore have the desired make up and will be the ones we adopt. Note also that the pot is inevitably empty when the players are at the preflop so $x_P := x_P(0) = 1/2$. Given that the pot has a value of P, we will in fact use Alice's following decision criteria:

Preflop: $\mathbb{E}[X] \geq x_P := \frac{1}{2}$;

Flop: $\mathbb{E}[X] \geq x_F(P) := \frac{3}{P+6}$;

Turn: $\mathbb{E}[X] \geq x_T(P) := \frac{2}{P+4}$;

River: $X \geq x_R(P) := \frac{1}{P+6}$.

We will now deal with the distinct variants of the game allowing us to better take into account the *Texas Hold'em Poker* game specifics. We provide strategies when the stakes are varied, when a blind is added or when there are more than 2 players.

4.3.2 Strategy with increasing stakes

In a game of Poker, stakes empirically appear to increase with each round of the game. We have already presented the optimal strategy with varied stakes in section 4.2.5. Let us call m_P, m_F, m_T and m_R the stakes at preflop, flop, turn and at the river respectively. These stakes are known in advance as being dependent on the pot. By adapting the rationale described in the previous section, for a pot that is worth a given value of P, the following criteria are easily obtained:

Preflop: $\mathbb{E}[X] \geq x_P := \frac{1}{2}$;

Flop: $\mathbb{E}[X] \geq x_F(P) := \frac{m_F + m_T + m_R}{P + 2 * (m_F + m_T + m_R)}$;

Turn: $\mathbb{E}[X] \geq x_T(P) := \frac{m_T + m_R}{P + 2 * (m_T + m_R)}$;

River: $X \geq x_R(P) := \frac{m_R}{P + 2 * m_R}$.

Alice's strategy is rather cautious. Intending to play until the last round, Alice chooses a relatively high level of cards to bet.

4.3.3 Strategy with a blind

In the game considered so far, players decide simultaneously whether they want to bet or not and they do not have a forced bet (blind). Let us consider a game where each player is forced to wager 1 chip at preflop every other game. This way, folding has an additional cost and the 2 player's roles are asymmetrical. So we isolate 2 cases depending on whether it is Bob or Alice who pays the blind.

1. Bob pays the blind: then Alice's strategy which is based on the fact that Bob will in any case bet, will not change.
2. Alice pays the blind: in this case Alice is forced to bet at the preflop. She then applies her strategy simply from the flop.

To conclude, Alice's strategy remains unchanged, apart from the fact that every other time, she has no choice but to bet at the preflop.

4.3.4 Strategy for a game with n players

We consider here a game where Alice has n opponents who have the same strategy as Bob. These n players bet with a probability of p and fold with a probability of $1-p$. Because she does not know p, Alice seeks a strategy based on the assumption that the other players always bet. All the same we assume that she will adapt her strategy to the number of players still in the game at each round.

Let us say that we are at the river with a pot of a value of P and a number n of players still in the game. Alice knows X, the probability of her hand beating another hand given the five community cards. Accordingly, discarding cards superposition, her probability of beating all n other players equates to X^n. As she wagers 1 chip in the hope of winning $P + n + 1$, her criterion of choice is obtained by the operation: $X^n \geq 1/(P + n + 1)$.

The same type of rationale can be applied to the various rounds of the game. The probability X of winning at the last round can simply be replaced by X^n and in the event of winning, the winnings can be adapted to the number of players n still in the game. Calling the different stakes at each round m_P, m_F, m_T and m_R, one obtains for a pot of a value of P and n players still in the game the following criteria:

Preflop: $$\mathbb{E}[X^n] \geq x_P := \frac{1}{2};$$

Flop: $$\mathbb{E}[X^n] \geq x_F(P) := \frac{m_F + m_T + m_R}{P + (n+1) * (m_F + m_T + m_R)};$$

Turn: $$\mathbb{E}[X^n] \geq x_T(P) := \frac{m_T + m_R}{P + (n+1) * (m_T + m_R)};$$

River: $$X^n \geq x_R(P) := \frac{m_R}{P + (n+1) * m_R}.$$

5 Digital tests

In his analysis, N. Alon calculates in a theoretical manner Alice's winning odds and the variance Y at each game where she uses her strategy. With these odds and variance, he deduces from the Central Limit Theorem the odds of Alice loosing after any random sufficiently high number *n* of games. In our more general game model, we cannot exactly calculate the odds and variance of this random variable *Y*. We have therefore chosen to perform digital simulations of virtual games in order to estimate them. These estimation techniques called the 'Monte Carlo' methods are represented in section 5.1.2

In order to implement Alice's strategy in a computer environment, we had to, at each round of the game, calculate her probability of winning $\mathbb{E}$ [X]. In order to calculate it at the river, at the turn and at the flop, we have simulated all the possible combinations of cards being revealed so that this probability can be calculated precisely. On the other hand, in order to calculate it during the preflop, when Alice only knows her two pocket cards, we have used tables of already calculated probabilities. These tables depend of course on the number of players and have been taken from [2] and [8]. In order to make Alice's strategy more realistic, we also present digital results where errors on the calculation of $\mathbb{E}$ [X] have been artificially introduced.

In this section, we first tackle the mathematical theorical justifications underlying our approach, then we present the digital results obtained.

5.1 Theoretical justification

5.1.1 The Central Limit Theorem

We lay out the manner in which, from Alice's winning odds and variance over a game, we can work out her odds of losing after a given number n of games. The result expressed had already been noticed and used by N. Alon [1].

We start by presenting (a version of) the central limit theorem.

Theorem 5.1 *Be M a real positive number and Y1, Y2, ... a suit of random variables independent of such laws in such a way that each Y_i satisfies $|Y_i| \le M$. Calling μ et σ^2 the odds and the variance of each X_i, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i = \mu, \quad \mathbb{P} - p.s.$$

In addition, we have

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}} \le z\right] = \Phi(z), \tag{5.1}$$

where Φ is the distribution function of normal law defined by :

$$\Phi(z) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}dt.$$

Let us assume that Y_i represents Alice winning at the i^{th} game. Then, $\sum_{i=1}^{n} Y_i$ is the sum of Alice's winnings over the first n games. Accordingly, Alice will be losing after *n* games if and only if $\sum_{i=1}^{n} Y_i \le 0$. As the random variables Y_i are independent and of the same law; we can apply the previous theorem and deduce the following result:

Proposition 5.1 Be *μ and σ^2 Alice's winning odds and variance at each game. The odds of Alice losing after a sufficiently high number n of games is in the order of* $\Phi(-\mu\sqrt{n}/\sigma)$.

In his analysis, N. Alon manages to calculate Alice's winning odds and variance at each game perfectly. In our more general game model, we cannot do this so we will obtain a digital approximation thanks to the Monte Carlo method.

5.1.2 The Monte Carlo methods

Given a specific strategy for Alice, we are seeking to estimate the odds μ and the variance σ^2 of her winning Y at each game. The Monte Carlo methods are based on the Central Limit Theorem stated above. Let us consider a pool of n games where Alice's winnings Y_i are available to us. Then, according to (5.1) in the theorm 5.1, we can estimate μ with

$$\hat{\mu}_N := \frac{1}{N}\sum_{i=1}^{N} Y_i .$$

Note that a standard estimator of Y 's variance is given by

$$\hat{\sigma}_N^2 := \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \frac{1}{N}\sum_{i=1}^{N} Y_i\right)^2 .$$

The idea is to use $\hat{\mu}_N$ and $\hat{\sigma}_N^2$ instead of μ and σ^2. We can demonstrate that the results stated in theorem 5.1 stay true when the variance σ^2 is replaced by its estimation $\hat{\sigma}_N^2$, see [4] for example. From this we conclude the following result:

Proposition 5.2 *Let us consider a pool of N poker games where, for each $i \leq N$, Y_i represents Alice's winnings at the i^{th} game. Then, the possibility of Alice losing after a sufficiently high number n of games is in the order of* $\Phi(-\hat{\mu}_N\sqrt{n}/\hat{\sigma}_N)$, *with*

$$\hat{\mu}_N := \frac{1}{N}\sum_{i=1}^{N} Y_i , \quad et \quad \hat{\sigma}_N^2 := \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \frac{1}{N}\sum_{i=1}^{N} Y_i\right)^2 .$$

5.2 Digital results

Here we present the digital results obtained in simulations of a pool of 30,000 poker games played between Alice and one or several random players. The strategies employed are those detailed in section 4.3. They were elaborated in the context of the opponent betting at all the rounds but we will test them in the context of the opponent betting randomly with a probability of *p*.

Alice's winning odds and variance are unknown and are therefore estimated with the Monte Carlo methods previously described. There is therefore a measuring error on the magnitude of these numbers which is absolutely controllable. The values provided below are not absolutely exact but the important thing is that their order of magnitude are completely valid.

In each variant of the game, the conclusion remains the same: for a sufficiently high number of games, strategy is a deciding factor in the outcome of a game.

5.2.1 The reference game

Let us consider first of all the reference game for which Alice's strategy has been presented in section 4.3.1, with a stake of 1 chip at each round. The following table provides Alice's estimated winning odds and variance at each game for various values of *p*. With proposition 5.2, we also calculate the odds of Alice losing after 50, 100 and 500 games.

P		1	0.9	0.8	0.7	0.6	0.5
Odds		0.32	0.35	0.36	0.36	0.33	0.30
Variance		6.7	5.2	3.9	2.8	2.0	1.4
% of chance for Alice to not be leading the game after...	50 games	18.7	15.7	12.2	8.7	5.3	2.5
	100 games	10.5	7.7	5.0	2.7	2.10^{-2}	2.10^{-3}
	500 games	0.25	0.07	0.02	9.10^{-4}	2.10^{-5}	3.10^{-8}

It's very clear that Alice's strategy gives her a considerable advantage. Figure 1, representing, for various values of p, the odds of Alice having lost money after n games, is also very telling. In the most unfavourable case where p = 1, the number of times when Alice is in a losing position is 1 in 10 after 100 games, less than 3 in a thousand after 500 games, and less than 4 in 100,000 after 1,000 games.

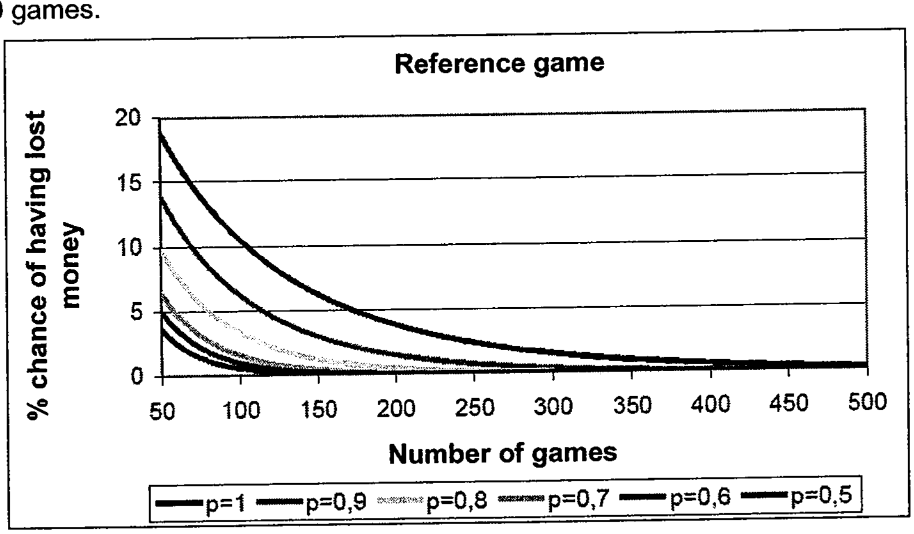

FIG. 1 - % chance for Alice to be in deficit vs the number of games

In a game of poker, each player has available to him or her the same initial number of chips and gets 'knocked-out' when he or she has no chips left. So in order to get knocked-out, a significant number of games have to be lost without winning too many of them. In this reference game there are exchanges of at most 4 chips at each round . After grading Alice's winnings distribution for a game, we have also estimated the odds of Alice losing given a specific initial number of chips. In the worst case scenario where p = 1, the results shown in the following table are very convincing. The higher the initial number of chips is, the more games Alice has to loose in order to be knocked-out. Therefore Alice gets knocked-out less often. We can observe for example that the number of times Alice looses is only 6 out of 1,000 with a fairly reasonable initial number of 50 chips.

Initial number of chips	10	25	50	100
Odds of Alice being knocked-out	24%	7%	0.6%	0.005%

5.2.2 Increase of stakes

Let us now consider the extended scenario where the stakes increase at each round of the game. Alice's strategy in this case is presented in 4.3.2. We are still making the assumption that

Bob bets with probability p and we present the digital results for cases where the stake is 1 at preflop, 2 at the flop, 4 at the turn and 8 at the river. The digital results are presented in the following table and in Figure 2.

P		1	0.9	0.8	0.7	0.6	0.5
Odds		1.5	1.3	1.1	1	0.8	0.7
Variance		61	50	41	33	25	19
% of chance for Alice to not be leading the game after...	50 games	8.9	9.9	10.8	11.2	11.8	12.0
	100 games	2.9	3.4	4.0	4.2	4.7	4.9
	500 games	10^{-3}	2.10^{-3}	4.10^{-3}	6.10^{-3}	9.10^{-3}	10^{-2}

In this version of the game, Alice has again a considerable advantage over her opponent. However, her winning variance is high because as many as 30 chips can be wagered in this game. Note that the winning odds decrease with *p*. Indeed Alice has a rather cautious strategy and the more often Bob folds, the less lucrative the games are for her, insomuch as the important stakes are at the end of a game. Even in the most unfavourable case analysed here, Alice's odds of losing are less than in the previous reference game. In the case where *p* = 1, the number of times when Alice is in a losing position is 3 in 100 after 100 games, 1 in 100,000 after 500 games, and less than 1 in a million after 650 games.

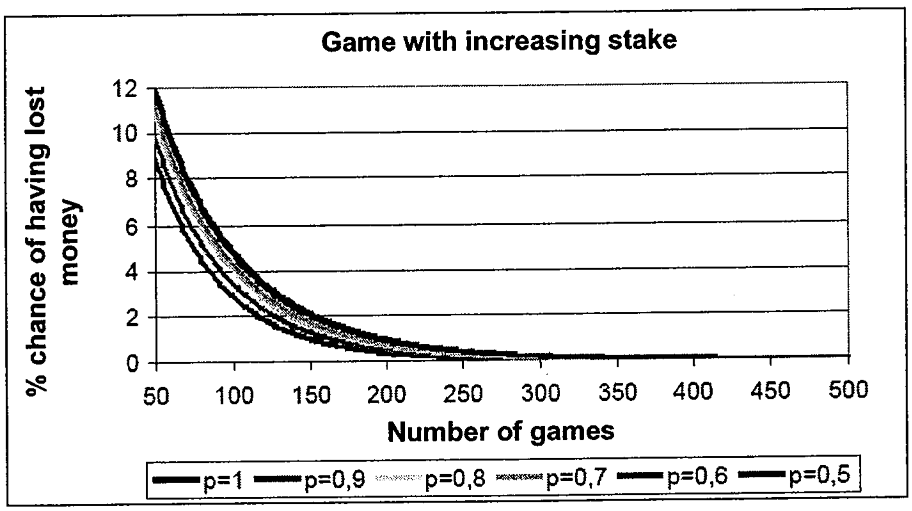

FIG. 2 - % chance for Alice to be in deficit vs number of games

5.2.3 Blind added

An important component in the rules of *Texas Hold'em Poker* is the use of forced stakes, a.k.a. 'the blind'. They force players to wager and put emphasis on their position around the table. Considering games with a forced stake alternating between the two players of 1 chip at the preflop, we have carried out a digital test of the strategy presented in section 4.3.3. In the following table, we show Alice's estimated winning odds and variance. As previously, the results are complemented by the odds of Alice not leading after n games, see Figure 3.

P	1	0.9	0.8	0.7	0.6	0.5
Odds	0.28	0.32	0.36	0.36	0.39	0.38

Variance		8.4	6.7	5.2	4.0	2.9	2.1
% of chance for Alice to not be leading the game after...	50 games	24.9	18.7	13.2	8.5	5.4	3.2
	100 games	16.9	10.4	5.7	2.6	1.1	0.4
	500 games	1.6	0.24	2.10^{-2}	7.10^{-4}	2.10^{-5}	2.10^{-7}

Unsurprisingly, Alice's performance is not as good in this game model as she sometimes has to wait for the second round of the game before she can fold despite the fact that she may have a bad hand. Like in the previous two games, strategy is nevertheless still the overiding factor in the outcome of a game. Indeed, in the most unfavourable case where $p = 1$, the number of times that Alice is in a losing position is 17 in 100 after 100 games, less than 2 in 100 after 500 games and less than 1 in a thousand after 1,000 games.

In order to compare this game with the reference game, we have also graded Alice's winnings distribution for a game and estimated the odds of Alice losing given an initial number of chips. In the worst case scenario where p = 1, the results are shown in the following table. We can observe for example that the number of times Alice looses in this instance is 3 in 100 for an initial number of 50 chips.

Initial number of chips	10	25	50	100
Odds of Alice being knocked-out	32%	15%	3%	0.01%

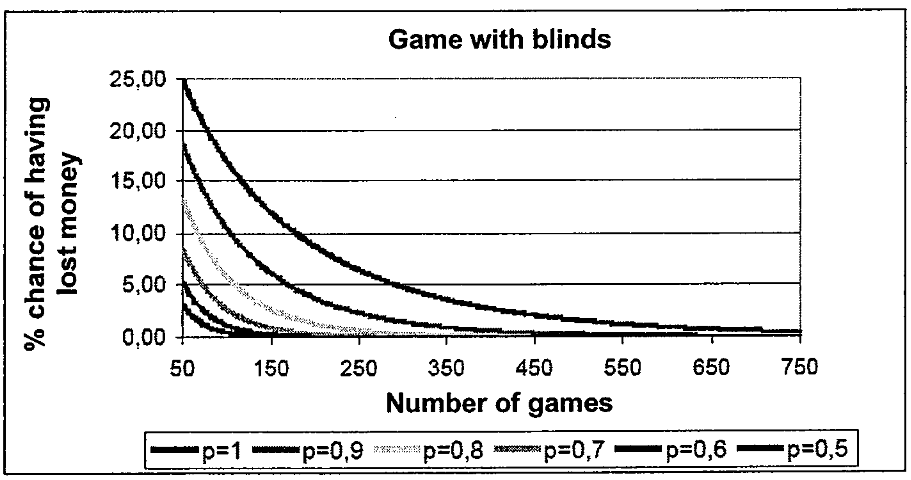

FIG. 3 - % chance for Alice to be in deficit vs number of games

Game where Alice does not have the precision of a computer

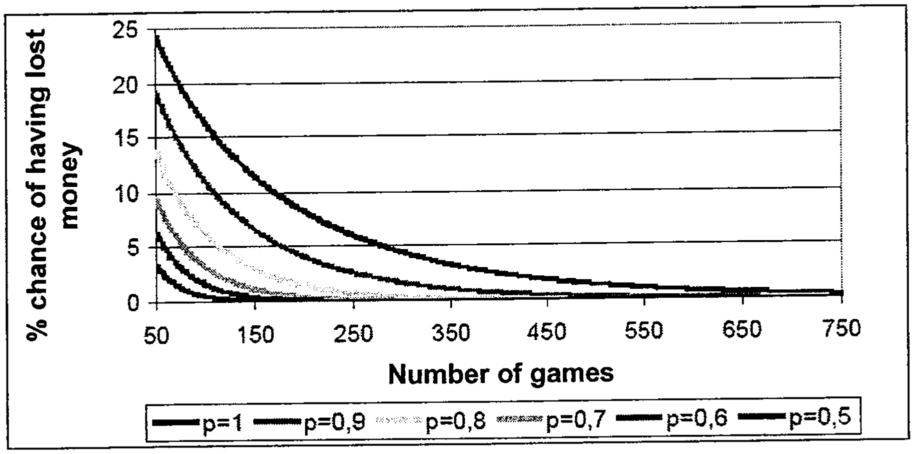

FIG. 4 - % chance for Alice to be in deficit vs number of games

5.2.4 Alice does not have the precision of a computer

In all the strategies employed until now, we always assume that Alice knows how to perfectly evaluate her hand. An experienced player certainly always has a precise idea of his or her 'hand in the pocket' potential, however it does not seem reasonable to expect a player to be able to evaluate his or her hand in such a precise manner. In order to make this analysis more realistic, we have artificially added a random error on the evaluation that Alice makes of her hand. We have mathematically added the odds of winning $\mathbb{E}$ [X] calculated by Alice with an independent centered normal law with a variance of 10^{-2}. In other words Alice now estimates her odds of winning and her approximation has a precision of 0.2, 95 times out of 100. For example, if Alice's odds of winning are 0.6, she will estimate them between 0.4 and 0.8 and will adapt her strategy to her estimation. The results obtained in this way are laid out in the following table and in Figure 4.

P		1	0.9	0.8	0.7	0.6	0.5
Odds		0.23	0.27	0.30	0.33	0.35	0.36
Variance		5.5	4.7	3.9	3.2	2.6	2.1
% of chance for Alice to not be leading the game after...	50 games	24.4	19.3	13.8	9.5	6.5	3.5
	100 games	16.3	11.0	6.2	3.2	1.7	0.5
	500 games	1.4	0.3	2.10^{-2}	3.10^{-3}	9.10^{-5}	5.10^{-7}

We arrive to the same conclusions as previously: even if Alice wins less often because she badly estimates her hand potential, her strategy remains dominant over that of her opponent. This way, in the most unfavourable case where $p = 1$, the number of times when Alice is in a losing position is 16 in 100 after 100 games, less than 2 in 100 after 500 games, and less than 1 in 1,000 after 1,000 games.

In this game model, we have also estimated Alice's odds of losing given an initial number of chips. In the worst case scenario where $p = 1$, the results are shown in the following table. In this instance, the number of times that Alice looses is 1 in 100 for an initial number of 50 chips.

Initial number of chips	10	25	50	100
Odds of Alice being knocked-out	28%	10%	1.3%	0.02%

5.2.5 A 4-player game

We now consider a 4-player game, where Alice plays against 3 players who have a random strategy characterised by *p*. Alice uses the strategy laid out in section 4.3.4. We have performed a digital estimation of Alice's winning odds and variance. In a game of 4 players, proposition 5.2 allows the calculation of Alice's odds of losing money after n games. The results obtained for various values of p are laid out in the following table and in Figure 5.

P		1	0.9	0.8	0.7	0.6	0.5
Odds		0.60	0.88	0.89	0.86	0.82	0.69
Variance		20	17	12	8	6	4
% of chance for Alice to not be leading the game after...	50 games	17.2	6.4	3.5	1.8	0.8	0.5
	100 games	9.1	1.5	0.5	0.2	3.10^{-2}	10^{-2}
	500 games	0.1	7.10^{-5}	5.10^{-7}	2.10^{-9}	7.10^{-13}	2.10^{-14}

Once again, the conclusion is similar: for a sufficiently high number of games played, the players' results are very clearly correlated with their respective strategies. Thus, in the most unfavourable case where p = 1, the number of times that Alice loses money is around 9 in 100 after 100 games, 1 in a thousand after 500 games, and 1 in 100,000 after 1,000 games.

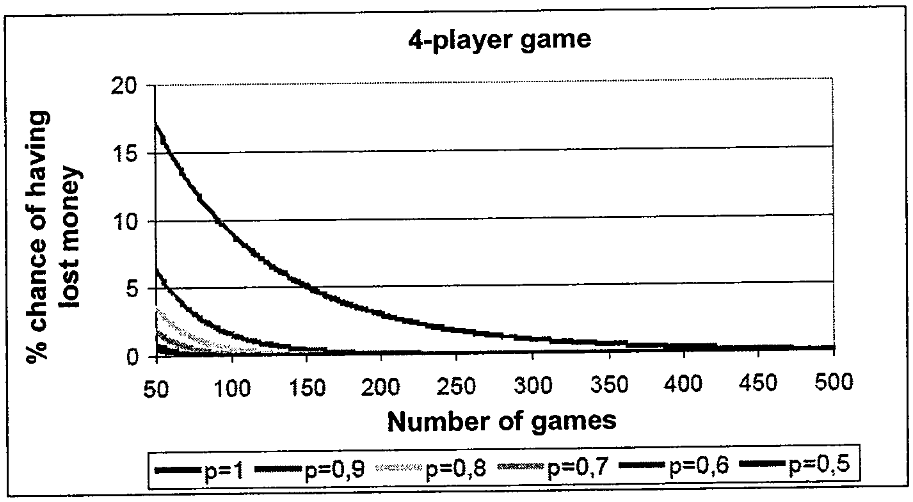

FIG. 5 - % chance for Alice to be in deficit Vs Number of games

Conclusions

Here we have analysed the influence of chance on the outcome of poker games between several players, one player having a dominant strategy over the others. In order to determine this dominant strategy, we have assumed that the other players were playing randomly and we have carried out a theoretical analysis over the expected winnings of a strategical player. Stemming from this analysis, we opted for a strategy which was not optimal but which was easy to understand.

In order to quantify the performances of a strategic player versus his opponents, we have performed a computer simulation of a pool of virtual poker games. This has enabled us to evaluate the winning odds and variance of a strategic player and to work out his or her chances of winning. We have considered game cases with 2 or 4 players, with or without blind, with constant or variable stakes. We also studied the case where the strategic player estimates his or her hand potential with little precision.

In all the game variants, the conclusion remains the same: for a sufficiently high number of consecutive games of *Texas Hold'em Poker*, the quality of the strategy employed has an overiding influence over the outcome of the game. Our conclusions are therefore similar to N. Alon's [1] but we have also dealt with broader game models. Furthermore these conclusions are completely consistent with the empirical observation that it is usually the same professional players who reach the final phases of Poker tournaments.

References

[1] Alon N. (2007). Poker, Chance and Skill. Preprint.

[2] Montmirel F. (2007). Poker Cadillac, *Fantaisium.*

[3] Cutler W. (1975). An optimal strategy for Pot-limit Poker, *The American Mathematical Monthly.*

[4] Fishman G.S. (1995). Monte Carlo concepts, algorithms and applications. *Springer series in Operation Research.*

[5] Von Neumann J. & O.Morgenstern (1944). Theory of Games and Economic Behaviour, *Princeton University Press.*

[6] Arrêté du 14 mai 2007 relatif à la règlementation des jeux dans les casinos (Decree of the 14th of May 2007 regarding the regulation of games in Casinos), (2007) Official *Journal of the French Republic.*

[7] Kelly J. M., Dhar Z. & T. Verbiest (2007). Poker and the Law : Is it a Game of Skill or Chance and Legally Does it Matter ? *Gaming Law Review* 11(3), 190-202. doi :10.1089/glr.2007.11309.

[8] Poker, rules and strategies. "http ://www.poker-regle/strategie.com/" rubrique "probabilites".

[9] Poker on Wikipedia. "http ://fr.wikipedia.org/wiki/Poker".

EXHIBIT D

Chance and Skill in Poker

Professor Abraham J. Wyner

April 17th, 2008

I. Biography of the Author

Professor Abraham (Adi) Wyner is Associate Professor of Statistics at the Wharton School of Business. He came to Wharton in 1999, from the University of California at Berkeley, where he was an Assistant Professor and a NSF Post-Doctoral Fellow. Dr. Wyner did his B.S. in Mathematics at Yale University where he won the Stanley Prize for excellence in Mathematics, heading west to complete his doctorate in Statistics on the West Coast, at Stanford University.

Professor Wyner's principle focus at Wharton has been research in Applied Probability, Information Theory and Statistical Learning. He has published more than 20 articles in leading journals in many different fields, including Statistics, Probability, Information Theory, Computer Science and Bio-Informatics. He has received many grants from the NSF, NIH and private industry. Professor Wyner has participated in numerous consulting projects in various businesses.

He was one the earliest consultants for TiVo, Inc, where he helped to develop personalization software. Dr. Wyner created some of the first on-line data summarization tools, while acting as CTO for Surfnotes, Inc. More recently, he has developed statistical analyses for banks and marketing research firms and has served as consultant to several law firms in Philadelphia, New York and Washington, D.C. In addition, he has served as statistical faculty advisor for the University Pennsylvania Law School. His interest in sports statistics has led to an ongoing collaboration with ESPN.com and "ESPN: the Magazine" where Dr. Wyner is the PI on the ESPN funded MLB player evaluation research project. He has served as faculty advisor to the Wharton Quant Club, numerous MBA cohorts and the Wharton Gaming club. For several years he taught an undergraduate course in Gaming that was so popular that over 1000 students competed for only 12 slots.

II. Is Poker predominantly a game of Skill?

In this consultation, I will address the question of whether poker (and more specifically Texas Hold'em Poker) is a game whose outcome is dependent more on skill than on chance, by evaluating two scientific articles where the issue has been analyzed in detail. One is an article by Professor Noga Alon, of Tel Aviv University (which is attached to this opinion as Annex A) and a second is essentially a follow up to Alon's article, written by Laure Elie and Romuald Elie of the University of Paris (which is attached to this opinion as Annex B). They have applied mathematical techniques to provide scientific evidence to the fact that poker is a game wherein winning is more dependent on skill than on chance.

III. ***Poker, Chance and Skill***, by Professor Noga Alon:

Noga Alon considers the game of "Texas Hold'Em" for which he provides a detailed and accurate description. Then he calculates probabilities for each type of hand and explains how knowledge of these probabilities is necessary in order to wager in a way that will maximize the expected winnings. This is his first intimation that a skilled player, who is able to calculate probabilities and use those calculations, will have an advantage over a player who cannot. One course of action, rejected by Alon, is to attempt to mathematically quantify the level of skill in a game. Instead, Alon constructs a simplified game of Texas Hold 'Em poker which he uses a model. The basic argument is that of *a fortiori*:

> *if it is possible to demonstrate that Skill predominates in simplified Texas Hold 'Em, than all the more so it will dominate for the actual game.*

Alon constructs several different single-betting-round games based on a "basic game" constructed to depend on the ranking property of poker. The games are as follows:

1. A two player game involving a beginner "Bob" who plays randomly against an advanced "Alice," who plays optimally.

2. An extension of the previous involving an "Improved" Bob against "Adapted" Alice, who adapts her advanced strategy to counter Bob.

3. A multiple player extension with Advanced Alice against multiple beginners in a ring game.

Alon shows that it is possible in these simple games to calculate exactly the strategy that Alice should play in order to maximize her expected winnings per round. Alon finds such a strategy for all three versions and then he calculates Alice's expected winnings per round and the variance of her winnings. He then applies the Central Limit Theorem for repeated independent events to calculate (for version 1) the approximate chance that Alice does not have more money than Bob after *n* rounds of play. As an added twist, he calculates the same probability with a blind bet instead.

For the simplest version, Alon shows that Alice's skill will dominate Bob's luck based approach. In fact, we have that:

- Although 7 out of 8 games end in a draw, if there is a winner, then Advanced Alice is 3 times more likely to win than Beginner Bob

- After 15 rounds of play, the chance that Alice is ahead is about 84%.

- After 150 rounds, the chance that Alice is ahead is about 99.9%.

To summarize, Alon accurately argues that:

- Knowledge of hand probabilities is a learned skill fundamental to determining and implementing an advanced strategy
- An advanced strategy will earn more than a strategy of an unskilled player with high probability in the short run
- An advanced strategy will earn more than a strategy of an unskilled player in the long run, with certainty.

So it is abundantly clear that in a simple game which pits and expert against a novice[1], the skilled player will dominate quickly. **Skill is the deciding and dominant factor.**

IV. Limitations and Extensions: ***Chance and Strategy in Poker.*** By Laure Elie and Romuald Elie.

The Alon analysis is of course limited to a basic simplified one round game of pseudo-poker. To conclude that poker itself is predominantly skill, one has to accept that the intricacies of actual poker will necessarily favor the skill factor, from which it follows, *a fortiori*, that real poker is predominantly skill. The argument is a heuristic, but it is compelling.

A second limitation is Alon's choice of players where Bob, who plays with basically no skill at all, challenges advanced player Alice. A more convincing argument would show that skill dominates the outcome of a game involving a highly skilled opponent against a player of modest abilities.

This challenge is met by the analysis in the article "*Chance and Stategy in Poker*" of Laure Elie and Romuald Elie, of the University of Paris. They build upon Alon's analysis extending the basic game to multiple round play, with pre-Flop, Flop, Turn, and River rounds, which follows the format of Texas Hold'Em itself. They also consider challengers who employ a range of strategies. Much of their article is devoted to developing the multiple-round game and calculating the optimal strategy for Alice. Since the game is too complex to calculate the expectation and variance of the each player's winnings, the instead simulate millions of rounds using the computer. This method, appropriately called "Monte Carlo" in the statistics literature, is an extremely effective way to approximate (to and desired level of accuracy) difficult to calculate probabilities, averages and variances.

The analysis presented in this article examines poker games involving blinds, increasing stakes and tournaments (i.e. "knock-out" games). In each, the optimal (or nearly optimal) player Alice is challenged by a range of opponents indexed by their probability p of calling/betting in a given round. The main conclusions are as follows:

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[1] *On the other hand, when two equally skilled players challenge each other the outcome is determined predominantly due to chance. This is true for all games, including athletic competitions. This is why a poker match involving the world's best players seems to be often decided by chance.*

- In the basic game, after only 50 rounds of play Alice has at worst less than a 20% chance of being behind even the most skilled challenger (*p*=1). After 500 rounds, this chance is less about ¼ of 1%.

- In a game with increasing stakes, Alice has at worst a 12% chance of behind after 50 rounds even against her most skilled challenger (*p*= ½). After 100 rounds the chance is less than 5%.

- In tournament style play, Alice has less than a 1/100 of 1% chance of being knocked out when each player starts with 100 chips. That chance increases to at most 32% when the starting stakes are only 10 chips.

- In a 4 player game, Alice has less than 20% chance of being behind after only 50 games even against 3 modestly skilled opponents (p=1). After 500 plays that chance is less than 1/10 of 1%.

The conclusion is very obvious. A skilled player will trounce lesser skilled opponents not only in the long-run, but also, with high probability, after what amounts to a session of only a couple of hours. Furthermore, in tournament play, where the number of rounds is not fixed, the skilled player has a decisive advantage even with modest initial stakes. **Skill is the dominant and decisive factor.**

V. Summary and Conclusion

Poker while simple enough to learn and play with only a short lesson, is extremely intricate and complex. A skilled player who is able to calculate correctly the probabilities of different hand configurations and is able to use that knowledge to bet and bluff appropriately has a substantial advantage over players without these skills. The two papers evaluated here ably demonstrate using mathematical analysis and computer simulation exactly how decisive that advantage is. The player who just "hopes to get the cards" will get them from time to time, but even after a single evening of play against a top player, he will be decisively beaten. **Skill dominates chance in poker.**